UNIQUE HISTORICAL DOCUMENTS OR JARNÍK'S MATHEMATICAL NOTEBOOKS FROM GÖTTINGEN

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Abstract

Among the artifacts deposited in the Archive of the Academy of Sciences of the Czech Republic are fourteen notebooks containing the lectures given by Karl Grandjot, Pavel Sergeevich Aleksandrov, Bartel Leendert van der Waerden and Emmy Amalie Noether in Göttingen in the 1920s. These unique and valuable notebooks were kept by Vojtěch Jarník, the future Czech university professor of mathematics, during his studies at the University in Göttingen. In this article, we attempt to give a basic characterization of Jarník's notebooks, to describe the historical background of their "birth" and to show their mathematical contents.

Keywords: Czech Mathematics, Göttingen University, Vojtěch Jarník, 20th Century.

[DOCUMENTOS HISTÓRICOS ÚNICOS OU OS CADERNOS MATEMÁTICOS DE JARNÍK ESCRITOS EM GÖTTINGEN]

Resumo

Entre os artefatos depositados no Arquivo da Academia de Ciências da República Tcheca, estão catorze cadernos contendo as palestras dadas em Göttingen por Karl Grandjot, Pavel Sergeevich Aleksandrov, Bartel Leendert van der Waerden e Emmy Amalie Noether, na década de 20 do século XX. Esses cadernos únicos e valiosos foram mantidos pelo tcheco Vojtěch Jarník, futuro professor universitário de matemática, durante os seus estudos na Universidade de Göttingen. Neste artigo, tentaremos dar uma caracterização básica dos cadernos de Jarník, para descrever o contexto histórico de seu "nascimento" e apresentar seus conteúdos matemáticos.

Palavras-chave: Matemática Tcheca, Universidade de Göttingen, Vojtěch Jarník, Séc. XX.
Mathematics in Göttingen

After the First World War, the University in Göttingen reached its academic peak: a high level of work prevailed not only in mathematics, but also in physics, chemistry, biology as well as in the social sciences and humanities. There was, in Göttingen, a vibrant scientific atmosphere owing to a large and revitalized academic community, which included gifted students, distinguished visitors from all over the world, and guest professors who came to present papers, give seminars or hold regular lectures.

In the second decade of the 20th century, the Göttingen mathematical community consisted of four ordinary professors, three or more extraordinary professors, some guest professors, several private docents, lecturers, senior and junior assistants, about two hundred undergraduate and graduate students and visitors. Among the members of the university staff we can find many outstanding mathematicians as for example: Felix Bernstein (1878–1956), Paul Bernays (1888–1977), Richard Courant (1888–1972), Gustav Herglotz (1881–1953), David Hilbert (1862–1943), Edmund Landau (1877–1938), Hermann Minkowski (1864–1909), Otto Neugebauer (1899–1990), Emmy Amalie Noether (1882–1935) and Hermann Weyl (1885–1955).

During the 1920s and at the beginning of 1930s many visiting professors spent some time in Göttingen in order to lecture or to collaborate with others there (for example Emil Artin (from Hamburg), Reinhold Baer (from Freiburg), Ruth Moufang (from Frankfurt), Richard von Mises (from Berlin), John von Neumann (from Berlin), George Polya (from Zurich), Oswald Veblen (from Princeton)). At the same time, many future outstanding mathematicians studied in Göttingen or worked there as Courant's, Hilbert's or Landau's assistants (for example Herbert Busemann, Max Deuring, Saunders MacLane, Gerhard Gentzen, Olga Taussky), in addition, a number of mathematicians obtained their **Habilitation** in mathematics, applied mathematics or mathematical physics at the University in Göttingen. Some of them continued their careers as **Privatdozenten**, extraordinary professors or ordinary professors there; others lectured at German universities as well as at European mathematical institutions or later moved on to positions in the USA.[1]

Vojtěch Jarník and his studies in Göttingen

From the 1870s onwards, the most talented and outstanding mathematicians and physicists from the Czech lands went abroad, enabled by government scholarships and funding, in order to extend and deepen their mathematical knowledge and skills. They travelled mainly to Germany, France or Italy and studied at the most prestigious mathematical centers of the period, in Berlin, Göttingen, Hamburg, Leipzig, Munich, Paris, Strasbourg, Milano, and Rome. There, they hoped to be more closely involved in the latest mathematical trends and methods and to become acquainted with the newest ideas in the field; they also hoped to be able to have their first papers accepted in respected journals and their first monographs issued by internationally known publishing houses; and finally, they

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[1] For information about the academic staff in Göttingen see [BN], [Nu] and [S].

sought out the most advanced education methods which they could bring back on their return to universities and polytechnics in the Czech lands. We will discuss the case of a young Czech mathematician Vojtěch Jarník (1897–1970).

Jarník after completing his studies at Charles University in Prague (1915–1919) started working as an assistant at the Czech Technical University in Brno. In 1921, he returned to Prague and became an assistant to Karel Petr (1868–1950), professor of mathematics at Charles University, under whose supervision he successfully defended his dissertation (1921, devoted to Bessel functions). In the 1920s, he spent three academic years in Göttingen, where he collaborated with E. Landau and E. Noether. From the 1920s, Jarník was continuously, except for the period of Nazi occupation when Czech institutions of higher education were forcibly closed, a member of the staff of Charles University. In 1925, he defended his **Habilitation** thesis (devoted to lattice points) and became an associate professor. In 1929, he was appointed an extraordinary professor and six years later he was appointed a full professor of Charles University. Jarník was interested in problems of lattice points, Diophantine approximations, geometry of numbers, set theory, topology, measure theory and the theory of integral. He was probably the first Czech mathematicians whose scientific results received a wide and lasting international response and continue to be cited to the present day.[2]

In 1923, Jarník left Prague and travelled to Göttingen where he studied and worked until February 1925. He was apparently most influenced by Landau, a distinguished specialist in mathematical analysis and number theory. (It should be mentioned that Jarník had studied analytic number theory and Landau's well-known works before his visit to Göttingen.) Once there, he regularly attended the lectures of Landau and Noether, which were well received by and inspiring for younger mathematicians. He also took part in their seminars and was in close touch with their students.

In the academic year 1927/1928 Jarník paid his second long-term visit to Göttingen and continued his collaboration with Landau.[3] He also regularly attended the lectures of K. Grandjot (1900–1979), P. S. Aleksandrov (1896–1982), B. L. van der Waerden Waerden (1903–1996) and E. Landau.[4] It should be noted that Landau deeply influenced Jarník's professional career and without any doubt he was Jarník's second important "mentor". Throughout Landau's life, they were in the close touch and often collaborated together.[5] It should be mentioned that Jarník had a bound collection of most of Landau's reprints.

From the second half of the twenties, Jarník was interested in number theory (especially in problems of lattice points) and theory of real functions. He wrote his articles in the Czech, German and French languages and published them mainly in Czech or German

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[2] More detailed information about Jarník's life and his mathematical achievements can be found in [BeNe1] and [No].

[3] Jarník's second stay was supported by the International Education Board from which he received a scholarship to spend one academic year in Göttingen to work on number theory with Landau. Jarník was strongly recommended and supported by Petr. For more information see [SS], pp. 293.

[4] Jarník chose Landau's lecture on higher analysis (summer semester 1927/1928).

[5] For more information about their collaboration see [BeNe1] and [No].

journals.

Thanks to his studies in Göttingen and his conversations with German mathematicians, and under their influence, Jarník finished more than ten works, some of which were published in Mathematische Zeitschrift, Mathematische Annalen and Annali di matematica pura ed applicata.[6] At the end of the twenties and during the thirties, Jarník continued his study of number theory and Diophantine approximations. He published his results in Mathematische Zeitschrift, Monatshefte für Mathematik und Physik, Tôhoku Mathematical Journal as well as in Czech journals.[7]

Jarník's notes of Göttingen mathematical lecture courses

Among the most special archival materials from this period deposited in the Archive of the Academy of Sciences of the Czech Republic are fourteen "notebooks" which contain the lectures of E. Noether, K. Grandjot, P. S. Aleksandrov and B. L. van der Waerden. These notebooks were kept by V. Jarník during his studies at Göttingen in the academic years 1923/1924, 1924/1925 and 1927/1928.

The notebooks were discovered by Jindřich Bečvář in 2004 when he was preparing an extensive monograph on the life and work of Vladimír Kořínek (1899–1981), a leading Czech algebraist of the 20th century.[8] Kořínek's unusually vast archival collection containing his personal, pedagogical and professional materials as well as some materials of his friends and colleagues from Charles University and the Czechoslovak Academy of Sciences is of special interest as it allows us to trace the development of mathematics in the Czech lands.

Although Jarník and Kořínek were good friends and colleagues, we are not able to explain how Jarník's notebooks came to be deposited in Kořínek's archival collection.

Specifically, at Göttingen, Jarník studied modern structural algebra under Noether in the academic year 1923/1924 and 1924/1925: the notebooks contain her lectures titled **Körpertheorie** (winter semester 1923/1924), **Invariantentheorie** (summer semester 1923/1924), **Gruppentheorie II** (winter semester 1924/1925), **Hyperkomplexe Zahlen und Gruppencharaktere** (winter semester 1927/1928); he studied the theory of numbers and modern structural algebra under van der Waerden in the academic year 1927/1928: the notebooks contain his lectures titled **Allgemeine Idealtheorie** (winter semester 1927/1928) and **Algebraische Zahlen** (summer semester 1927/1928); he also studied modern algebra under Grandjot in the winter semester in 1927/1928: the notebooks contain his lectures titled **Algebra II (Galoissche Theorie)**; and finally he studied analysis under Aleksandrov in the summer semester 1927/1928: the notebook contains his lectures titled **Punktmengen und reelle Funktionen**. It is not without interest that Jarník attended lectures predominantly on modern algebra and very rarely those on number theory and analysis,

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[6] For more information about Jarník's publication activities see [BeNe1] and [No].

[7] For more information see the bibliography of Jarník's scientific works published in [No].

[8] Kohoutová Z., Bečvář J.: **Vladimír Kořínek (1899–1981)** (in Czech), Edition History of Mathematics, Volume 27, Research Center for the History of Sciences and Humanities, Prague, 2005.

although these two topics represent his main mathematical subjects.

Basic characterization of Jarník's notebooks

Jarník's notes were kept in rectangular exercise books (16,4 × 20,6 cm) each with a hard black cover; they have been preserved in an amazingly good condition. His German notes are carefully written in blue ink; almost everything is legible. They have few grammatical and syntactic mistakes, almost no corrections and contain very few inaccuracies.[9] Each notebook has 120 pages, usually completely filled with notes. On the interior page of the cover, Jarník's Göttingen address is written (in the academic year 1927/1928 – Dr. V. Jarník, Göttingen, Bühlstrasse 28).

Jarník's notebooks give us a record of Göttingen's mathematical lectures and seminars, which were famous in the Czech lands before the Second World War due to the high professional level. They also provide information on mathematics and teaching in Göttingen, information not generally known even in Germany. Most importantly, they were written by an excellent Czech mathematician who possessed an acute understanding of the material being presented. Since similar documents from that time are rare, they are a unique contribution to our understanding of this period, and should be interesting not only for mathematicians, but also for historians, linguists and anyone wanting to learn something about mathematics in the first half of the 20th century.[10]

Grandjot's lectures "Algebra II"

Jarník's first four notebooks are devoted to Grandjot's lectures **Algebra II (Galoissche Theorie)**. The first 116 pages of Jarník's first notebook contain Grandjot's lectures divided into six paragraphs.[11] The remaining four pages are blank. The first lecture was written on November 1, 1927; the last one written on November 29, 1927. The first 117 pages of Jarník's second notebook contain seven paragraphs of Grandjot's lectures.[12] The remaining three pages are blank. The first lecture was delivered on December 2, 1927; the last one was delivered on December 20, 1927. The first 118 pages of Jarník's third notebook contain four paragraphs of Grandjot's lectures.[13] The remaining two pages are blank. The date of the first lecture is not mentioned; the second lecture was probably delivered on January 17, 1928. In this notebook Jarník did not precisely write down the dates of the lectures as he did in the first two notebooks.[14] The first 53 pages of Jarník's

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[9] Only the records of Noether's lectures are not quite so clear.

[10] For more information see [BeNe1].

[11] The paragraphs are titled: **I. Körper und Integritätsbereiche**, **II. Teilbarkeit und Zerlegungssätze**, **III. Grundbegriffe der Gruppentheorie**, **IV. Restklassenringe. Charakteristik**, **V. Nullstellen von Polynomen**, **VI. Endliche algebraische Erweiterungen**.

[12] The last part of the paragraph **VI. Endliche algebraische Erweiterungen**, **VII. Normalteiler und Faktorgruppen**, **VIII. Der Hauptsatz der Galoisschen Theorie**, **IX. Endlicher Körper**, **X. Der Satz vom primitiven Element. Theorie der Gleichungen**, **XI. Ergänzung zum Hauptsatz**, **XII. Aufslösbare Körper**.

[13] The last part of the paragraph **XII. Aufslösbare Körper**, **XIII. Entscheidung über Irreduzibilität und Galoissche Gruppen**, **XIV. Irreduzible Gleichungen von Primzahlgrad**, **XV. Körper von reellem Bau**.

[14] The January 24, 1928, is the last date written down in the notebook.

fourth notebook contain one paragraph of Grandjot's lectures.[15] The remaining 67 pages are blank. In this notebook Jarník did not note the date of the lectures. We suppose that Grandjot lectured to the end of the winter semester 1927/1928, that is, to the end of February 1928. The two-hours lectures were given twice a week.[16]

Jarník's notes are divided into fifteen parts which include the standard facts presented today in a basic university course on algebra for mathematicians. The definitions of basic terms, theorems with their complete proofs as well as remarks, comments and exercises are added. Almost no bibliographical comments are given.[17]

Aleksandrov's lectures "Punktmengen und reelle Funktionen"

The fifth notebook is devoted to a series of Aleksandrov's lectures titled **Punktmengen und reelle Funktionen**; it is one of the most important and interesting part of Jarník's mathematical notebooks. The first 107 pages of the notebook contain Aleksandrov's lectures, the last nine pages of the notebook contain a short part of van der Waerden's lecture titled **Algebraische Zahlen** which continues in Jarník's ninth notebook, and the remaining four pages are blank.

In this notebook Jarník did not write down the dates of the lectures. From his inscription we know that he attended Aleksandrov's lectures in Göttingen in the summer semester 1927/1928. We do not know precisely the period in which the course was delivered and how many lectures it consisted of. However, Aleksandrov was in Göttingen between June 4 and August 4, 1928.[18]

In [BeNe1], on the basis of our detailed study, we tried to give an overview of the ideas and results presented in the course. We gave the definitions, theorems as well as remarks, and in several places we added comments on the methods of proofs. We tried to trace the origin of the results in order to place the material in historical context. Several quotations and bibliographical comments illustrate the fact that the first quarter of the 20th century was a fascinating period for the development of descriptive set theory, real analysis and point-set topology.[19]

Aleksandrov's course on point sets and real functions has several remarkable features which we summarize as follows.

The course

– reflects the state-of-the-art in important parts of these fields of mathematics as it was in 1928

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[15] The last part of the paragraph **XV. Körper von reellem Bau**.

[16] The "University calendar" for the winter semester 1927/1928, p. 23, provides the following information: **Algebra II (Galoissche Theorie), Di. Fr. 4–6. Dr. Grandjot.**

[17] K. Grandjot was not a specialist in algebra, resp. Galois theory. His lectures **Algebra II** were not the top at the University in Göttingen and we will not analyze them.

[18] For more information see [BeNe1] and Aleksandrov's letters to F. Hausdorff which are available in **Nachlass Hausdorff**, Kapsel 61, Universitäts und Landesbibliothek Bonn, Handschriftenabteilung.

[19] See [BeNe1], pp. 91–110.

– is very **modern** in the sense that practically all the results presented were less than thirty years old, the majority were less than ten years old, and some had not yet been published

– centers around several excellent results established by P. S. Aleksandrov himself as well as by his collaborators and colleagues such as Paul Samuilovich Urysohn (1898–1924) and Mikhail Yakovlevich Suslin (1894–1919); in particular, we have in mind the proof of the Continuum Hypothesis for Borel sets, the A-operation, properties of analytic sets, topologically complete spaces and the role of zero-dimensional spaces

– includes principal results and concepts that today have become the standard facts taught in basic university courses for mathematicians

– has a strong topological flavour even though it provides results in the context of the real line, Euclidean spaces and metric spaces; the notion of a topological space is not even mentioned; this reflects the fact that Aleksandrov was one of the leading architects in the construction of topology as a mathematical subject

– uses mathematical language which is fully set-theoretical, and a style of exposition of results and their proofs that is similar to the contemporary way of presenting mathematics: the main difference being that the use of transfinite numbers and transfinite induction is much more frequent in comparison with present style, this also applies to the use of continued fractions

– shows only a few differences in notation from today: $x \subset A$, $A + B$, AB, $A - B$, ΣA_n, ΠA_n were later replaced by $x \in A$, $A \cup B$, $A \cap B$, $A \setminus B$, $\cup A_n$, $\cap A_n$, respectively, the empty set is, unlike our $\emptyset$, written as 0, $\lim_{n=\infty}$ unlike our $\lim_{n\to\infty}$; also the distinction between f (= a function) and f (x) (= the value of f at the point x) is not usually respected.

    In summary, the course was delivered by a distinguished expert whose impact on contemporary mathematics is still felt today.

    Jarník's notes can be naturally divided into three parts: **Punktmengen** (47 pages), **Bairesche Funktionen und Borelsche Mengen** (24 pages) and **Suslinschen Mengen** or **A-Mengen** (36 pages).[20]

    The lectures are presented within the frame of Euclidean spaces or metric spaces. Completeness of metric spaces and topologically equivalent metrics are discussed first, then zero-dimensional spaces are treated. Here, topological ideas come into the picture quite strongly. The exposition is based on the article published by Aleksandrov and Urysohn in Mathematische Annalen in 1928, four years after Urysohn's tragic death. Then the transfinite induction definition of Borel sets is given and the role of G_δ-sets is emphasized

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[20] The transcription of Jarník's notebook can be found in [BeNe1], pp. 51–90.

(the set of continuity points of an arbitrary function, the importance for measure theory). Next, extension properties of continuous (or homeomorphic) mappings are discussed and topological completeness, based on important results of Aleksandrov and Hausdorff, is studied. It is shown, using Hausdorff's approach, that G_δ-sets are nothing else than topologically complete spaces. Baire functions and Borel sets, as well as their interplay, are treated. In particular, the existence of α-th class Baire functions not belonging to any lower class is proved. Special attention is paid to the notion of analytically representable functions extensively studied by Baire and Lebesgue during the first decade of the 20th century. The last section is devoted to analytic (or Suslin) sets, discovered ten years before the lecture course was delivered. Analytic sets, as a result of the A-operation, are introduced and their set-theoretic properties are established. Their relation to Borel sets is especially analyzed; specifically, Borel sets are characterized as those analytic sets having an analytic complement. The Continuum Hypothesis for uncountable analytic sets is proved. This generalizes Aleksandrov's famous result for Borel sets from 1916.[21] Finally, a short survey of some, at that time, new results from descriptive set theory is given.

Van der Waerden's lectures "Allgemeine Idealtheorie"

Jarník's sixth notebook contains 116 pages of van der Waerden's lectures titled **Allgemeine Idealtheorie**, the last 4 pages are blank. The first 54 pages of the seventh notebook contain van der Waerden's lectures titled **Allgemeine Idealtheorie**, a scheme with the "classification of rings" was then added to the notebook (it was not written by Jarník); the 2 following pages are blank and the next 60 pages of the notebook contain a part of van der Waerden's lectures titled **Algebraische Zahlen** which continues in Jarník's fifth, eighth and ninth notebooks.[22] In the first half of the sixth notebook Jarník wrote down the dates of the lectures,[23] in the second half of the sixth notebook and in the seventh notebook he did not write any date of the lectures. From his inscription we know that he attended van der Waerden's lectures titled **Allgemeine Idealtheorie** in Göttingen in the winter semester 1927/1928. We conjecture that the period in which the course was delivered was from the beginning of November 1927 to the end of February 1928. The one-hour lectures were given twice a week.[24]

Jarník's notes can be naturally divided into five parts: **Kapitel I** (64 pages, 8 paragraphs),[25] **Kapitel II – Körpertheorie** (46 pages, 9 paragraphs),[26] **Kapitel III – Idealtheo-**

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[21] Alexandroff P.: **Sur la puissance des ensembles mesurables** B, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Série Mathématiques, Paris 162(1916), pp. 323–325. This is the first paper by Aleksandrov and a germ of the A-operation, which opened the way to Suslin sets, is presented there.

[22] Two remaining pages of the seventh book are blank.

[23] November 4, 1927 is the first recorded date (written on the first page), December 12, 1927 is the last one (written on the page 93).

[24] The "University calendar" for the winter semester 1927/1928, p. 23, provides the following information: **Allgemeine Idealtheorie, Mo. Fr. 6–7. Dr. van der Waerden**.

[25] The paragraphs are titled: **Einleitung, Gruppen, Ringe, Quotientenkörper, Polynomring, Restklassenring, Weiteres über Polynomringe Divisionsalgorithmus, Idealtheorie der Euklidischen Ringe**.

[26] The paragraphs are titled: **Primkörper, Einfache Körpererweiterungen, Lineare Abhängigkeit, Endliche und algebraische Körpererweiterungen, Galoissche Erweiterungen, Algebraisch abgeschlossene Körper,**

rie in Polynombereichen (18 pages, 4 paragraphs),[27] **Kapitel IV – Allgemeine Idealtheorie** (23 pages, 7 paragraphs)[28] and **Kapitel V – Ganze algebraische Grössen** (19 pages, 3 paragraphs).[29]

We note that in the summer semester of 1926, van der Waerden was in Hamburg where he studied with Emil Artin (1898–1962), Erich Hecke (1887–1947) and Otto Schreier (1901–1929). He attended Artin's well-regarded courses on algebra[30] and collected notes and ideas with the intention of collaborating with Artin on a book planned for Springer-Verlag's "Yellow Series".[31] But when Artin saw the first drafts of van der Waerden's text, he suggested that van der Waerden writes the book alone, without Artin's participation. This text became van der Waerden's famous textbook **Moderne Algebra I.** and **Moderne Algebra II.**[32] On February 26, 1927, van der Waerden was appointed a university lecturer at the University of Göttingen (that is "Privatdocent" according to the German academic system). He spent the academic years 1927 and 1928 there, lectured on modern algebra and number theory, prepared his papers and collaborated with Noether.

It seems that van der Waerden's lectures **Allgemeine Idealtheorie** (one of the most interesting part of Jarník's mathematical notebooks) are almost a full prototype for his monograph **Moderne Algebra I.** Now, we intend to analyze the notebook's contents and compare them with van der Waerden's monograph which was very highly esteemed and translated into foreign languages and stimulated several generations to learn modern algebra and its applications.[33]

Transzendente Erweiterungskörper, Algebraische Funktionen, Erweiterungen erster und zweiter Art (separable und inseparable Erweiterungen).

[27] The paragraphs are titled: **Der Hilbertsche Basissatz, Algebraische Mannigfaltigkeiten, Nullstellen-theorie der Primideale, Geometrische Deutung beliebiger Ideale**.

[28] The paragraphs are titled: **Basissatz und Teilerkettensatz, Der Zerlegungssatz, Idealprodukte und Quotienten, Geometrische Anwendungen des Zerlegungssatzes, Die Eindeutigkeitssätze, Theorie der teilerfremden Ideale, Die Vielfachenkettensatz**.

[29] The paragraphs are titled: **Moduln in Bezug auf einen Ring, Theorie der ganzen Grössen, Idealtheorie der ganz-abgeschlossenen Ringe**.

[30] In [FTW], p. 138, it is written that van der Waerden was an assistant at the University of Hamburg during the summer semester 1926/1927.

[31] It also turns out that van der Waerden found some ideas in Noether's lectures, seminars and works, as well, for example **Zur Theorie der Polynomideale und Resultanten**, Mathematische Annalen 88(1923), pp. 53–79; **Eliminationstheorie und allgemeine Idealtheorie**, Mathematische Annalen 90(1923), pp. 229–261; **Eliminationstheorie und Idealtheorie**, Jahresbericht der Deutschen Mathematiker-Vereinigung 33(1924), pp. 116–120.

[32] For more information about van der Waerden's inspirations see [Wa1]. See also **Moderne Algebra. B. L. van der Waerden. Unter Benützung von Vorlesungen von E. Artin und E. Noether**, I. Theil, Die Grundlehren der mathematischen Wissenschaften, Band 33, Springer, Berlin, 1930, VIII + 243 pages, ..., 9th edition, 1993, XI + 272 pages; **Moderne Algebra. B. L. van der Waerden**, II. Theil, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 34, Springer, Berlin, 1931, VII + 216 pages, ..., 6th edition, 1955, XI + 300 pages; Russian translations (1947, 1976, 1979), English translations (1949, 1970, 1991, 2003), Portuguese translation (1954), Chinese translation (1964).

[33] The team consists of Jindřich Bečvář and Martina Bečvářová.

Van der Waerden's lectures "Algebraische Zahlen"

Jarník's seventh and eighth notebooks, as well as the part of the ninth and sixth notebooks contain the lecture course titled **Algebraische Zahlen** delivered by van der Waerden during the summer semester 1927/1928.[34] The last 60 pages of the seventh notebook contain 8 paragraphs of van der Waerden's lectures;[35] the first lecture was delivered on April 30, 1928. The lectures following this are not dated. We assume that van der Waerden lectured to the end of the summer semester 1927/1928, that is, to the end of July 1928. Van den Waerden's lectures continue in Jarník's eighth notebook. There are 7 paragraphs (61 pages),[36] 2 pages are blank, followed by two other paragraphs (24 pages)[37] and again 2 blank pages.[38] The first 21 pages of Jarník's nineth notebook contain two paragraphs of van der Waerden's lectures.[39] Two blank pages follow and then, there are 73 pages which are devoted to mathematical analysis: first, 3 paragraphs on Phragmén-Lindelöf theorem (the paragraphs have the numbers 7, 8, and 9),[40] then, chapter IV. titled **Wachstum und**

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[34] Probably the lectures were inspired by the books of H. M. Weber: **Lehrbuch der Algebra. Band II**, F. Vieweg und Sohn, Braunschweig, 1896, XIV + 796 pages, and **Elliptische Funktionen und algebraische Zahlen**, F. Vieweg und Sohn, Braunschweig, 1891, XIII + 504 pages. The second volume contains four paragraphs: **Gruppen** (pp. 1–160), **Lineare Gruppen** (pp. 161–347), **Anwendungen der Gruppentheorie** (pp. 349–550) and **Algebraische Zahlen** (pp. 551–844). The third volume contains six paragraphs: **Analytischer Teil** (pp. 1–317), **Quadratische Körper** (pp. 319–410), **Komplexe Multiplikation** (pp. 411–560), **Klassenkörper** (pp. 561–620), **Algebraische Funktionen** (pp. 621–707) and **Tabellen** (pp. 709–726). Weber's textbook was the last major algebra textbook that summarizes the results of algebra at the end of the 19th century. It brought new concepts of nascent structural algebra and it became for more than 30 years the standard text for algebraic studies. It inspired new generations of mathematicians to study structural algebra and also contributed to the development and consolidation of modern algebraic terminology. It was finally replaced by van der Waerden's **Moderne Algebra** (Volume I. and II., 1930 and 1931). The other source of van der Waerden's inspiration could be R. K. E. Fricke: **Lehrbuch der Algebra verfaßt mit Benutzung von Heinrich Webers gleichnamigen Buche. Band III, Algebraische Zahlen**, F. Vieweg und Sohn, Braunschweig, 1928, VIII + 506 pages. The volume contains two chapters: **Allgemeine Theorie der Zahlkörper** (pp. 2–189) and **Ausführungen über besondere Zahlkörper** (pp. 190–502). In the introduction to the first volume of **Moderne Algebra**, van der Waerden wrote that E. Artin's, W. Blaschke's and O. Schreier's seminar on the theory of ideals (Hamburg, 1926) and E. Nother's lectures on theory of group and hypercomplex numbers (Göttingen, 1924/1925 and 1927/1928) were the main headspring of his ideas.

[35] The paragraphs are titled: **1. Algebraische und ganze algebraische Zahlen**, **2. Zahlkörper**, **3. Die Hauptordnung eines Zahlkörpers**, **4. Einheiten, Faktorzerlegung**, **5. Einfachste Eigenschaften der Ideale**, **Gebrochene Ideale und Faktorzerlegung**, **Die Gruppe der gebrochenen Ideale**, **8. Normen der Ideale**.

[36] The paragraphs are titled: **9. Struktur des Restklassenringes**, **10. Quadratische Körper**, **9ª . Galoissche Körper**, **11. Der kubische Körper** ($\Gamma^3\sqrt{q}$), **12. Der Diskriminantensatz**, **13. Kreiskörper der I^h-ten Einheitswurzeln**, **14. Beziehungen zwischen Idealen verschiedener Körper**.

[37] The paragraphs are titled: **15. Zerlegungs- und Trägheitsgruppe**, **16. Zusammenhang zwischen Zerlegungen der Ideale im Ober- und Unterkörper**.

[38] At the end of the eighth Jarník's notebook, there are two paragraphs titled **1. Komplexe** (4 pages) and **2. Orientierung** (3 pages) of an unnamed lecture. We do not know precisely the period in which they were delivered nor by whom.

[39] The paragraphs are titled: **17. Die Klassenzahl**, **18. Der Einheitensatz**.

[40] The first six paragraphs are missing.

Phragmén-Lindelöf principle is a form of the maximum-modulus theorem for holomorphic functions that permits weaker hypotheses compensated with extra conditions. The original version is due to Lars Edvard Phragmén (1863–1937) and Ernst Leonard Lindelöf (1870–1946) (see L. E. Phragmén, E. L. Lindelöf: **Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le**

Koeffizienten [Growth and coefficients, 3 paragraphs] and chapter V. titled **Das Picardschen Problem** [Picard's problem, 7 paragraphs].[41] We do not know the name of the lecturer, the title of the lecture, the period in which they were delivered and how many lectures it consisted of.[42] The end of van der Waerden's lecture **Algebraische Zahlen** (9 pages) was written by Jarník on the last nine pages of his fifth notebook.[43]

      Jarník's notes include the standard facts presented today in a basic university course on abstract algebra for mathematicians. The definitions of basic terms, theorems with their basic proofs as well as short remarks, comments and exercises are added. Almost no bibliographical comments are given. In the future, we will focus on van der Waerden's lectures **Algebraische Zahlen** and we will analyze them.

Four Noether's lectures

      From 1920 to 1926, Emmy Noether devoted herself to developing the theory of mathematical rings, ideals and modules, the theory of group representations, the invariant theory of finite groups, the elimination theory. From 1927 to 1935, she focused on hypercomplex numbers, the representation theory of groups and algebras, the theory of central simple algebras, the structure theory of associative algebras, the single arithmetic theory of modules and ideals in rings satisfying ascending chain conditions.

      Noether delivered the lecture course titled **Gruppen Theorie II** in the winter semester 1924/1925, that is, from the beginning of November 1924 to the end of February

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voisinage d'un point singulier, Acta Mathematica 31(1908), pp. 381–406).

From complex function theory it is known that if a function f is continuous on the closure of a bounded region D and holomorphic on D, then the maximum of |f | is attained on the boundary of D. If, however, a domain D is not bounded, then this is no longer true. W. Rudin in **Real and complex analysis** (McGraw-Hill Book Company, New York, 1987, p. 256) writes: *"A method developed by Phragmén and Lindelöf makes it possible to prove theorems of the following kind: If f is holomorphic on D and if |f | < g, where g(z) → ∞ "slowly" as z → ∞ in D (just what "slowly" means depends on D), then f is actually bounded in D, and this usually implies further conclusions about f , by the maximum modulus theorem"*.

In the 1920s and the 1930s, **Phragmén-Lindelöf type theorems** and their applications was a modern topic (see for example M. Riesz: **Sur le principe de Phragmén-Lindelöf**, Proceedings of the Cambridge Philosophical Society 20(1920), pp. 205–209, correction 21(1921), No. 6).

[41] **Picard's theorem** describes the behavior of a holomorphic function in a neighborhood of a singular point. It reads as follows: *"If a holomorphic function f has an essential singularity at a point w, then on any neighborhood of w, the function f takes on all complex values, with at most a single exception, infinitely often"*. This theorem was proved by Charles Émile Picard (1856–1941) in 1879 in the form of the so called **Picard's little theorem**: *"Every entire function whose range omits two points must be a constant function"*. (See **Sur les fonctions analytiques uniformes dans le voisinage d'un point singulier essentiel**, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Série Mathématiques, Paris 88(1879), pp. 745–747; **Sur une propriété des fonctions entières**, ibid., pp. 1024–1027). For more details on **Picard's theorem** see for example S. G. Krantz: **Handbook of Complex Variables**, Birkhäuser, Boston, 1999, p. 140, W. Rudin: **Real and complex analysis**, McGraw-Hill Book Company, New York, 1987, pp. 331–332, and R. Remmert: **Classical topics in complex function theory**, Springer-Verlag, New York, 1998.

[42] According to the "University calendar" for the winter semester 1927/1928 and the summer semester 1927/1928 we know that mathematical analysis was lectured on by R. Courant, K. Grandjot, G. Herglotz, A. Ostrowski, F. Walther, etc. In the winter semester 1927/1928, E. Landau was on the vacation.

[43] The text is under the heading: **v. d. Waerden, Algebraische Zahlen, Göttingen, Sommersemestr 1928. (Schluss).**

1925.[44] These lectures are recorded in Jarník's tenth notebook which contains 56 pages of notes (the other pages are blank) and includes no date of any of the lectures, no indication of chapters, subchapters, sections and subsections. The notes show that Noether's lectures were difficult to follow, understand and record, but they must have been very inspiring as they contained Noether's latest results and the ideas that she had just worked out.[45]

Jarník's eleventh notebook contains Noether's lectures titled **Hyperkomplexe Zahlen und Gruppencharaktere**.[46] The first lecture was delivered on November 11, 1927; the last one on January 14, 1928.[47] Jarník's records contain three parts: **Historischer Uberblick** (5 pages), **Gruppen** (66 pages) and **Modul- und Darstellungstheorie** (12 pages). The remaining pages are blank. As a curiosity we note that in the notebook, there is a pink blotter paper with a caricature of the lecturer.

Jarník's twelfth notebook contains the lectures titled **Invariantentheorie** delivered by Noether in the summer semester 1923/1924.[48] Jarník's records are divided into four parts: **Einführung** (11 pages), **Formale Processe zur Bildung der Invarianten** (12 pages), **Symbolische Darstellung** (3 pages) and **Der Endlichkeitssatz** (47 pages).[49] Jarník did not note the date of any of the lectures delivered and his records are not easy to read.

The 43 pages of Jarník's twelfth notebook contain the lectures titled **Körpertheorie**.[50] We do not know the name of the lecturer, the period in which the course was delivered and how many lectures it consisted of because Jarník did not include any dates. From the style of Jarník's notes and its mathematical content, we suppose that the

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[44] The "University calendar" for the winter semester 1924/1925, p. 20, provides the following information: **Gruppentheorie, Mi. So. 9–11. Prof. Noether**.

[45] See for exemple E. Noether (N. Jacobson (ed.)): **Gesammelte Abhandlungen**, Springer-Verlag, Berlin, New York, 1983, VIII + 777 pages.

[46] Noether developed her most important ideas on hypercomplex numbers and groups of characters in her articles **Hyperkomplexe Größen und Darstellungstheorie**, Mathematische Zeitschrift 30(1929), pp. 641–692; **Hyperkomplexe Grössen und Darstellungstheorie in arithmetischer Auffassung**, Atti Congresso Bologna, Volume 2, Bologna, 1930, pp. 71–73; **Hyperkomplexe Systeme in ihren Beziehungen zur kommutativen Algebra und Zahlentheorie**, Verhandlungen Kongress Zürich, Volume 1, Zürich, 1932, pp. 189–194; **Der Endlichkeitssatz der Invarianten endlicher linearer Gruppen der Charakteristik** p, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1926, pp. 28–35; etc. See also E. Noether (N. Jacobson (ed.)): **Gesammelte Abhandlungen**, Springer-Verlag, Berlin, New York, 1983, VIII + 777 pages. For a detailed analysis of the historical context and comments on the development of hypercomplex numbers and representation theory see T. Hawkins: **Hypercomplex numbers, Lie groups, and the creation of group representation theory**, Archive for History of Exact Sciences 8(1972), pp. 243–287.

[47] The "University calendar" for the winter semester 1927/1928, p. 23, provides the following information: **Hyperkomplexe Größen und Gruppencharaktere, So. 11–1. Prof. Emmy Noether**.

[48] The "University calendar" for the summer semester 1923/1924, p. 20, provides the following information: **Invariantentheorie, Mi. So. 11–1. Prof. Emmy Noether**.

[49] It is likely that some of lectures were based on Noether's results published in her papers **Zur Invariantentheorie der Formen von** n **Variabeln**, Jahresbericht der Deutschen Mathematiker-Vereinigung 19(1910), pp. 101–104; **Der Endlichkeitssatz der Invarianten endlicher Gruppen**, Mathematische Annalen 77(1915), pp. 89–92; **Über ganze rationale Darstellung der Invarianten eines Systems von beliebig vielen Grundformen**, Mathematische Annalen 77(1915), pp. 93–102; **Invarianten beliebiger Differentialausdrücke**, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1918, pp. 37–44; **Invariante Variationsprobleme**, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1918, pp. 235–257. See also E. Noether (N. Jacobson (ed.)): **Gesammelte Abhandlungen**, Springer-Verlag, Berlin, New York, 1983, VIII + 777 pages.

[50] The last two pages of the notebook are blank.

lectures could have been delivered by Noether in the winter semester 1923/1924.[51] Unfortunately, Jarník's records are difficult to read; they are incomplete and disorganized.[52]

Janík's notebooks of Noether's lectures are difficult to read, contain many crossed out, corrections, inaccuracies and omissions. They are more than terrible to read and to analyze. Only native German mathematician specialised in modern algebra and number theory could analyze them and find some new or interesting moments showing the development of Noether's mathematical ideas and the style of her lecturing.

Jarník's last two notebooks on algebra and analysis

The lectures on **Gruppentheorie** and **Körpertheorie** continue in Jarník's thirteenth notebook which can be divided into five parts: **Gruppe** (20 pages), **Der Jordan-Höldersche Kompositionssatz** (23 pages), **Körpertheorie** (22 pages), **Direktes Durchschnitt und direktes Produkt** (36 pages).[53] Jarník's records omit the titles and dates of the lectures as well as who gave them.

The lecture on **Körpertheorie** continues in Jarník's fourteenth notebook (38 pages), followed by 15 blank pages. Starting from the back of the notebook, one can find 16 pages of a lecture titled **Hardyschen Satz** which was delivered on July 31, 1924. It should be emphasized that the lectures contained results which were less than fifteen years old. Unfortunately, the name of lecturer was not written.[54] But, we should note that this theme was very close to Jarník's interests.[55] Following this lecture are 3 pages with a vague and unintelligible mathematical text.[56]

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[51] The "University calendar" for the winter semester 1923/1924, p. 21, provides the following information: **Übungen (Vorträge zur Körpertheorie), Mo. 6–7. Prof. Emmy Noether.**

[52] It seems that some of lectures were based on Noether's results contained in her works **Rationale Funktionenkörper**, Jahresbericht der Deutschen Mathematiker-Vereinigung 22(1913), pp. 316–319; **Körper und Systeme rationaler Funktionen**, Mathematische Annalen 76(1915), pp. 161–196; **Die arithmetische Theorie der algebraischen Funktionen einer Veränderlichen, in ihrer Beziehung zu den übrigen Theorien und zu der Zahlkörpertheorie**, Jahresbericht der Deutschen Mathematiker-Vereinigung 28(1919), pp. 182–203; **Idealtheorie in Ringbereichen**, Mathematische Annalen 83(1921) pp. 24–66; **Module in nichtkommutativen Bereichen, insbesondere aus Differential- und Differenzenausdrücken**, Mathematische Zeitschrift 8(1920), pp. 1–35 (with W. Schmeidler).

[53] The remaining 19 pages are blank.

[54] The **Hardy-Littlewood Tauberian theorem** relates the asymptotics of the partial sums of a series and asymptotics of its Abel summation. The theorem reads as follow: *"Suppose $a_n \geq 0$, $\sum_{n=0}^{\infty} a_n e^{-ny} \sim \frac{1}{y}$ as $y \to 0_+$. Then $\sum_{n \leq x} a_n \sim x$ as $x \to \infty$"*. For more information see G. H. Hardy: **Ramanujan: Twelve lectures on subjects suggested by his life and work**, 3rd edition, Chelsea, New York, 1999, pp. 34–35.

Sometimes, the theorem is quoted in the equivalent form: *"Suppose $a_n \geq 0$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x}$ as $x \uparrow 1$, then as n tends to ∞, $\sum_{k=0}^{n} a_k \sim n$"*. For more information see E. C. Titchmarsh: **The theory of functions**, 2nd edition, Oxford University Press, Oxford, 1939, p. 226.

In 1914, the theorem was proved by Godfrey Harold Hardy (1877–1947) and John Edensor Littlewood (1885–1977) (**Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive**, Proceedings of the London Mathematical Society 13(1914), pp. 174–191; see also **Contributions to the arithmetic theory of series**, Proceedings of the London Mathematical Society 12(1913), No. 1, pp. 411–478).

[55] See K. Grandjot, V. Jarník, E. Landau, J. E. Littlewood: **Bestimmung einer absoluten Konstanten aus der Theorie der trigonometrischen Reihen**, Annali di Matematica Pura ed Applicata 6(1929), No. 4, pp. 1–7.

[56] The title of this part is **Satz von Lerch**. See Lerch's article **O stanovení součinitelů v mocninném rozvoji funkce ζ(s)** [On the determination of coefficients in the power series expansion of function ζ(s)], Časopis pro

Conclusion

Jarník's mathematical notebooks could be interesting for anyone who wants to understand how modern mathematics grow out of, nourish, diffuse over the world and give inspirations for the creation of new disciplines and futher development of mathematics.

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