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WHAT MAKES A PYTHAGOREAN PYTHAGOREAN?

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Abstract

The present paper characterizes the Pythagoreans as those thinkers that maintain that the *lógos* of the universe is revealed by a mathematical theory of proportion. Part of what is thereby revealed, according to the Pythagoreans, is the transmigration of the soul and the need to organize the mesocosm (society) in a manner isomorphic to the macrocosm (universe). It is also suggested that the organization of Pythagorean societies may have been the inspiration for Plato's doctrine of the Divided Line.

Keywords: Mathematics and Philosophy; Pythagoreanism; Theory of Proportion; Doctrine of the Divided Line.

Resumo

O presente trabalho caracteriza os pitagóricos como os pensadores que mantêm que o *lógos* do universo é revelado por uma teoria matemática de proporção. Parte do que é revelado, segundo os pitagóricos, é a transmigração da alma e a necessidade de organizar o mesocosmo (a sociedade) em uma maneira isomórfica ao macrocosmo (o universo). Sugere-se também que a organização de sociedades pitagóricas poderia ter servido como inspiração para a doutrina platônica da Linha Dividida.

Palavras-chave: Matemática e Filosofia; Pitagorismo; Teoria de Proporção; Doutrina da Linha Dividida.

Pythagoras, perchance because he left us no written account of his thought, has been a polemical figure both in the History of Mathematics and in the History of Philosophy. For some, he was no more than a shaman or a fanatical mystic, while, for others, he was the principal forerunner of the scientific worldview. The same dichotomy carries over to the Pythagoreans, a rather diverse conglomeration[1] of "schools" of thought, dispersed

[1] See, for example, Huffman (1993).

throughout the ancient world and related, somehow, to Pythagoras himself. I say "somehow" because the exact relation has not been clearly delineated in the literature on this topic. It will be the chief burden of the present paper to suggest a criterion for determining what it meant to be a Pythagorean. In so doing, we will be led to consider the organization of Pythagorean societies and this, in turn, will suggest that this organization was a model for Plato's theory of the Divided Line.

A Coherent Picture of Pythagorean Thought

The Pythagoreans, in general, are known for two apparently independent philosophical theses, which we may summarize in the following way:

1. All is number, or, more properly, all is number and harmony.
2. The soul is immortal and, after separation from the body, is reborn in another body.

It is with respect to these two theses that the eminent historian of ancient thought, F. M. Cornford (1922, p. 137)[2] opines that

> ... in the sixth and fifth centuries B.C., two different and radically opposed systems of thought were elaborated within the Pythagorean school. They may be called respectively the mystical system and the scientific. All current accounts of Pythagoreanism known to me attempt to combine the traits of both systems in one composite picture, which naturally fails to hold together.

Indeed, from the point of view of Nineteenth Century mathematics, one might be hard pressed to see how mysticism and science could hang together in a coherent whole. Nevertheless, from the ancient point of view, there was no conflict. Further, it should be mentioned that by characterizing Thesis 2 as "mystical", Cornford is, deliberately or not, prejudging its nature. Although this thesis originated in (or at least came to Pythagoras by way of) the Orphics and/or other mystic groups, there is nothing about it that is inherently mystical, and, in fact, the Pythagoreans gave rational arguments for its acceptance.

Still, it is Thesis 1 that gives Pythagoreanism its special character, as we shall see presently. The expression "all is number" clearly puts Pythagoras in the tradition of the Ionian Physicists, for, instead of Thales' water, Anaximandar's unlimited and Anaximenes' air, Pythagoras promotes number as the *arché* of the universe, that is, as the basic substance from which all else derives. Even so, we must remember that Pythagoras is reputed to have made a new start in philosophy. He did so by claiming that it is not enough to determine the *arché* of the universe; we must go beyond this to discover its *lógos*.

The primordial meaning of the Greek word *lógos* is "word", but it was also used in myriad other associated ways. In the present context it meant something like "that which gives sense to the world" or the "intelligibility" of the world. For the Pythagoreans, this search for *lógos* was subsumed in the addendum to Thesis 1, "and harmony". It is harmony that holds the various numbers together in a consistent whole and thus makes the resultant whole intelligible to human reason.

How are we, then, to conceive of harmony? It is well known that by harmony, or music, the Pythagoreans referred to the theory of ratios and proportions among (positive

[2] I wish to thank my colleague Josildo José Barbosa da Silva for making this article available to me.

whole) numbers. The ancients claimed that it was Pythagoras himself who discovered that the musical consonances were determined by ratios of whole numbers: the octave has a ratio of 2:1, the fourth that of 4:3 and the fifth that of 3:2. Thus, the basic[3] musical consonances are determined by the first four (positive whole) numbers, {1, 2, 3, 4}, which was called the *tetractys* and used in other philosophical speculations about the universe.

Thus, for Pythagoras, it was the mathematics that revealed the hidden structure of the universe. More specifically, it was the mathematical theory of ratios and proportions that revealed the *lógos* of the universe, thereby making it intelligible to human reason. Consequently, mathematical knowledge was sacred knowledge and, thus, not fully publicized. Since it was sacred knowledge, it was reserved for those that had undergone the appropriate initiation ceremonies and, in so doing, had made themselves worthy of receiving it.

The link with mathematics is underscored by the very name used to designate these initiates: *mathematikoí*. In fact, the name comes from a verb that originally meant "to know", so that the *mathematikoí* were "the learned ones" or, perhaps, "students". It was indeed amongst the very Pythagoreans that the name "mathematics" lost its more general sense of "knowledge" to take on the specific meaning of the mathematical disciplines of arithmetic (study of numbers), music (ratio and proportion), geometry (figures) and astronomy (figures in motion) [4]. Thus, for the Pythagoreans, mathematics was knowledge *par excellence*.

We are thus lead to a first approximation of what it meant to be a Pythagorean: basically, that Thesis 2 was a result of Thesis 1. We can flesh this out a bit as follows. For the Pythagoreans, mathematics was the *lógos* of the universe and, thus, revealed to man the sacred structure of the whole world, including the relation of mankind to the divine. Fundamental to this relation is the awareness of the immortality of the soul and its transmigration into other bodies at the death of the actual body. Thus, there is an extremely close, coherent relation between mathematics[5], which is rational, but not scientific (in the modern meaning of the term), and religion, which is also rational, though not without mystical overtones (which themselves are justified by the mathematics). Further, the sacred – and, thus, restricted– nature of mathematical knowledge is used to structure society according to the epistemological attainments of its members so that the structure of society (the mesocosm) parallels the structure of the universe (the macrocosm). At some point[6], this parallelism is extended to very the structure of the human being (the microcosm).

Thus, we may say that the diverse Pythagorean schools are the same – that is, they are Pythagorean – to the extent that they accept both Thesis 1 above and that Thesis 2 is a consequence of Thesis 1, as fleshed out in the preceding paragraph. They would be different to the extent that they disagreed in the exact way that the articulation between

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[3] There were other consonances in Greek music, notably the tone and the semitone, which were also explained in terms of numerical ratios. Since these may be considered as secondary, however, it is not necessary to consider them here.

[4] See, for example, Heath (1981).

[5] For more details on how this relation was thematized, see, for example, Fossa (2006).

[6] Certainly by the time of Plato, if not earlier.

Thesis 1 and Thesis 2 was made, or, perhaps more importantly, to the extent that their mathematical base differed.

It will behoove us to consider more closely the structure of Pythagorean societies. Before doing so, however, it will be useful to investigate the mathematical base of Pythagorean doctrines.

Theories of Proportion

As we have already seen, Pythagoras' original insight regarding mathematics as *lógos* seems to have been the recognition that musical consonances are structured by ratios among positive whole numbers. It is, in fact, ratio and proportion that account for the intelligibility of the universe, thus making it accessible to human reason. Even etymology confirms this conclusion, for, taken over from the Greek *lógos*, the Latin word *ratio* has the double meaning of "reason" (rational account, reckoning) and mathematical "ratio".

In the second generation of Pythagoreans, we see an increasing concern with mathematical means, especially the arithmetic, geometric and subaltern (later called harmonic) means. Archytas, in particular, is credited with the mathematical development of this theory, but fragments from other Pythagoreans, like Philolaus, evidence the application of mathematical means to diverse areas of philosophy, *e.g.*, ethics.

By the third generation of Pythagoreans, a (mini-)crisis developed with the discovery of incommensurability, which rendered invalid many of the proofs in the Pythagorean theory of proportion. The crisis was quickly overcome, however, by a new theory of proportions due to Eudoxus, a member (and perhaps cofounder) of Plato's Academy[7]. Plato himself, possibly in collaboration with other members of the Academy, developed an elaborate theory of the Divided Line[8] (utilizing the geometric mean), which he used to structure his metaphysics, while his student, Aristotle, generalized this to the Doubly Divided Line (utilizing continued proportion), which fell apart into two pairs of opposites, that is, the Fourfold, so characteristic of his thought[9].

The number theoretic character of Pythagorean mathematics is further highlighted by the persistent interest that it showed in figurate numbers throughout the long history of Pythagoreanism and their continued interest in proportion theory is shown by the later development of the doctrine of the ten mathematical means and by development of mathematical scales for (phenomenal) music theory. Even "sacred geometry", when it develops, is either subservient to arithmetic, as it is in Plato, or elaborated in later, non-Pythagorean schools.

All this evidence leads us to refine our conception of what it means to be a Pythagorean by limiting the mathematical base to the theory of proportion. Thus, a Pythagorean would be someone who asserts that the theory of proportion reveals the *lógos* of the universe to be something like Thesis 2 above with the resultant structuring of society according to the epistemological attainments of its members. The differences among the diverse schools would be due to the particular theory of proportion that was adopted.

[7] See Fossa & Erickson (2005).
[8] See Erickson & Fossa (2006).
[9] See Fossa (2007).

Pythagoras, for example, had a simple theory of proportion based on multiplicative relations, embedded, however, in an additive calculus. The theory of Archytas revolved about the aforementioned three mathematical means and that of Plato on the new theory of Eudoxus and the Divided Line.

This refinement of our understanding of Pythagoreanism has an important consequence. Como Fossa (2010) has shown, the theory of proportion was one of the earliest parts of mathematics to have developed and, therefore, carried with it appreciable connections with traditional knowledge. Thus, the Pythagoreans were, in fact, thinkers who embraced traditional values, as is shown by Pythagoras himself in his acceptance of Orphic principles. Ironically, however, new developments in mathematics challenged the Pythagoreans to make corresponding changes in their philosophy. Plato recognizes this explicitly in his desire to create new myths for the new times. Nevertheless, his new myths are imbued with traditional values.

The Structure of Pythagorean Societies

The Greek societies which came under the government of the Pythagoreans were divided, as already mentioned, into various classes. We will now turn our attention to an investigation of these classes.

The major division, of course, was between those who belonged to the Brotherhood and those who did not. In order to join the Brotherhood, it was necessary to undergo certain rites of initiation. It was reported in antiquity that one of the major rites was that of observing silence for five years. This was reputedly to insure that the prospective member had the self control and discretion not to reveal the sacred doctrine to those who were not appropriately prepared for receiving it. Further, the members renounced all their private property and goods and resided together, following various daily regimens and practicing certain virtues. Within the Brotherhood, there was an administrative council called the *politikoí*, consisting of the most advanced members of the group. Naturally, there must have been some variation on this basic model from place to place and from time to time.

According to Iamblichus (1987), a third century (A.D.) neoplatonic philosopher, the bulk of the Pythagoreans was called *cenobites*, because of their common life together. These were subdivided into two groups, the *mathematikoí* and the *akousmatikoí* (hearers). The latter group, by much the larger of the two subdivisions, was clearly, despite Iamblichus description to the contrary, not an integral part of the Brotherhood and probably did not share its common life. Rather, it consisted of people who frequented lectures open to the public and perhaps made more or less token payments or voluntary contributions to the Brotherhood without relinquishing their total wealth to it. They may have considered themselves Pythagoreans, but Iamblichus himself indicates that the *mathematikoí* did not so consider them. In any case, Iamblichus (1987, p. 77) goes on to say that

> The philosophy of the Hearers consisted in lectures without demonstrations or conferences or arguments, merely directing something to be done in a certain way, unquestioningly, preserving them as so many divine dogmas, non-discussable, and which they promised not to reveal, esteeming as most wise those who more than others retained them.

The *mathematikoí*, or learned ones, in contrast, were privy to mathematical demonstrations and their applications to philosophy.

The rest of the society, who we may designate by the term *hoi polloí*, the unruly masses, had no special contact with the Pythagoreans and had to be disciplined, for their own good, by civil authority.

Schematically, then, the classes in a Pythagorean society take the following form

politikoí

cenobites $\begin{cases} \text{mathematikoí} \\ \text{akousmatikoí} \end{cases}$

hoi polloí

where the hierarchy goes from top to bottom.

We can now compare two different Pythagorean schools, that of Pythagoras and that of Plato, for example, in the following manner:

<table>
<tr><td rowspan="2">members of
the Brotherhood</td><td>polítikoí</td></tr>
<tr><td>mathematikoí</td></tr>
<tr><td rowspan="2">non-members</td><td>akousmatikoí</td></tr>
<tr><td>hoi polloí</td></tr>
<tr><td colspan="2">Pythagoras'school</td></tr>
<tr><td colspan="2">proportion as
multiplicative constant</td></tr>
</table>

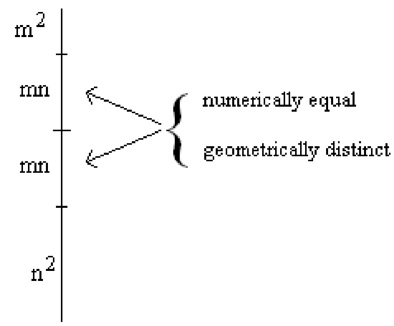
<table>
<tr><td rowspan="2">esoterics</td><td>members of
the Academy</td></tr>
<tr><td>students of
the Academy</td></tr>
<tr><td rowspan="2">exoterics</td><td>readers of
the Dialogues</td></tr>
<tr><td>hoi polloí</td></tr>
<tr><td colspan="2">Plato's school</td></tr>
<tr><td colspan="2">Doctrine of the Divided Line</td></tr>
</table>

The diagram above clearly shows that the structures of the two Pythagorean schools are isomorphic and that they differ in their mathematical base. We may represent this difference as a difference in the content of Thesis 1 above.

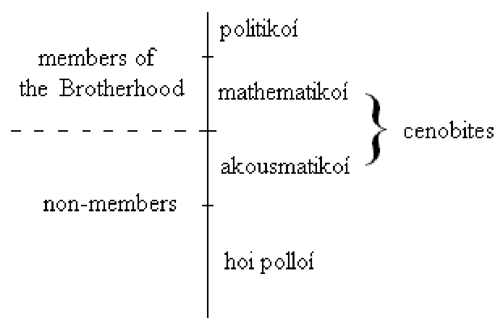
A Possible Origin of the Divided Line

Now that we have characterized the Pythagoreans as those that hold that the transmigration of the soul and the mesocosm-macrocosm isomorphism are consequences of the theory of proportion, we are led to a surprising insight, to wit, the very doctrine of the Divided Line may have been inspired by the organization of the Pythagorean societies as explained above.

To see this, we recall that the Divided Line is a line segment divided once, whose parts are then divided in the same ratio. From the number theoretic point of view, the result is a line whose extremes are perfect squares and whose middle parts are the geometric mean of these extremes, as in the following diagram:



Thus, the two middle parts are numerically equal, but geometrically distinct (congruent). Qualitatively, the middle parts are the same from one point of view, but different from another. This is the exact structure found in my schematization of Pythagorean societies, diagramed above. To make this even clearer, the scheme can be recast as an explicit Divided Line in the following manner:



From one point of view, the *mathematikoí* and the *akousmatikoí* are the same, because they both philosophize about the Pythagorean way of life. From another point of view, however, they are different, because the former have access to the mathematics and, thus, to the *lógos*, whereas the latter do not.

At first sight, the suggestion that Plato got the germ of his idea about the Divided Line from the structure of Pythagorean societies seems a bit farfetched. Nevertheless, when we recall that the doctrine of the Divided Line is found in the *Republic*, which treats of the microcosm in terms of the mesocosm, and that it was exactly while writing this dialogue that Plato became a Pythagorean, the suggestion becomes much less fanciful.

Conclusion

In conclusion, I would like to reflect a bit on the importance of the present characterization of the Pythagoreans for the History of Mathematics.

It is, of course, interesting for intellectual history to be able to conceptualize diverse groups as parts of a larger movement and to be able to pinpoint the common thread that gives them an underlying unity, while, at the same time, being able to sort out the various strands of thought that make up the mosaic of their differences.

It may be just as interesting, however, to step back and view mathematics as a social institution, thereby identifying the roles it plays, or has played, in different cultures. Some of these roles have become commonplaces, such as its role as an organizing principal in modern science and technology. Others, however, are just beginning to obtain recognition, as in the present case, where we see mathematics playing the role of an organizing principal of philosophical thought, stretching back, in fact, to the beginnings of metaphysics.

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