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ON THE ANCESTRY OF Z. P. DIENES'S THEORY OF MATHEMATICS EDUCATION

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Resumo

Z. P. Dienes, um educador matemático influente nos meados do século passado, desenvolveu uma importante teoria de educação matemática. Não obstante, não explicitou as pressuposições filosóficas da sua teoria. O presente artigo mostra que a teoria da educação matemática de Dienes se assenta sobre uma compreensão intuicionista do que seja a matemática e isto, por sua vez, mostra que o intuicionismo teve uma grande influência no desenvolvimento do construtivismo.

Abstract

Z. P. Dienes, an influential mathematics educator of the middle of the last century, developed an important theory of mathematics education. Nevertheless, he did not explicate the philosophical presuppositions of his theory. The present article shows that Dienes's theory of mathematics education rests on an intuitionist understanding of mathematics and this, in turn, shows that intuitionism was an important early influence on constructivism.

Zoltan Paul Dienes was born in Budapest, Hungary, in 1916. He later studied mathematics and became a very influential mathematics educator, following the general lines of Piagetian cognitive psychology. Although he traveled widely and assumed academic posts in various countries, much of his most influential work was done in England. In fact, the present work will review Dienes's theory of mathematics education as it was elaborated and presented in the 1960's and 1970's, the period in which his influence was at its greatest. We may also note that many Brazilian mathematics educators were directly influenced by Dienes[1], during the 1970's and 1980's. He currently resides in Canada and is active as a poet. His autobiography, *Memoirs of a Maverick Mathematician* was published in 1998.

During the above mentioned period, Dienes elaborated his theory of mathematics education, which centered on the use of manipulative materials in student-oriented, (re)discovery type activities. Since he was setting out this theory for the mathematics education community, he did not worry about making his philosophical presuppositions and

[1] I thank Antônio José ("Bigode") Lopes for this information.

sources explicit. As we shall see shortly, it was quickly realized that one of the influences on Dienes was intuitionism. Nevertheless, his ancestry was seen to be a hodgepodge of conflicting influences, in which intuitionism was but a minor player. In the present work, we will attempt to show that, in fact, intuitionism was the dominant influence on Dienes and that all his main themes are consonant with an intuitionist perspective on mathematics.[2]

The Initial Evidence

In the preface to Dienes's *An Experimental Study of Mathematics Learning*, J. S. Bruner writes

> *I have tried to find ancestry for Dr. Dienes's effort. One of the ancestral branches is surely the intuitive approach to mathematics that is usually lumped under the heading 'intuitionism'. The other branch is very likely the movements in education usually associated with the names of Frobel and Montessori.* (Bruner, "preface" to Dienes, 1963, p. 1.)

More recently, William Bart has entertained this topic and concluded that

> *Dienes, being an intuitionist mathematician, adheres to an eclectic philosophical position which has Kantian and Cartesian overtones, borrowing from each of the three categories of realism, conceptualism, and nominalism.* (Bart, 1970, p. 375.)

Eventually Bart (1970, p. 375) throws in pragmatism for good measure.

Both Bruner and Bart are correct regarding Dienes's intuitionism. Although he often seems to precind from philosophy in order the better to do psychology, only the most inattentive reader will fail to notice his intuitionist bias.[3] Moreover Dienes explicitly acknowledges his intuitionist viewpoint in "Sulla Definizione dei Gradi di Rigore" and in *Concept Formation and Personality*, while in *Building Up Mathematics* he invokes intuitionism's conceptual cousin and biggest bugaboo, formalism, in the following terms:

> *That is why this volume is called* Building Up Mathematics. *It is meant to send cold shivers down the spines of those who believe that mathematics is based on logic. In the author's view mathematics is based on experience; it is the crystallization of relationships into a beautifully regular structure, distilled from our actual contacts with the real world. Logic consists in reflecting on how this and other structures function, and I find it difficult to see how it is possible to reflect on something that is not there yet.* (Dienes, 1960, p. 11.)

[2] For more information see Fossa (1998).
[3] See, for example, Dienes (1959b) or Dienes (1964b).

It is difficult, however, to see any close family relationship in the intellectual physiognomy of some of the other proposed ancestors. In particular, Frobel was an absolute idealist, whereas Dienes would be a critical idealist (because of his intuitionism). Further, Dienes himself viewed Montessori as having taken some useful first steps toward reforming mathematics education, but without any theoretical basis.[4]

Again, Bart (1970, p. 370) claims that Dienes has pragmatic tendencies because he (Dienes) is concerned with the "relevance and the applicability of his techniques and concepts" and because Dienes affirms that

> *What we cannot say about any theory is that it is true, neither can we say that it is false. If it works, we use it; if it does not, we reject it.* (Dienes, 1964b, p. 18.)

Now, it would indeed be a *rara avis* of a mathematics educator who was blithely unconcerned with the applicability of his/her theory. Further, Dienes, in the above citation, is explaining scientific method along what has become quite traditional lines. Hence, neither of these claims serves to characterize Dienes's position as a pragmatic stance. In fact, the major thesis of pragmatism is that the analytic/synthetic distinction is a highly fuzzy one, but neither of the claims made above address this thesis. Nonetheless, Bart (1970, p. 370) also claims that Dienes's basic definitions are all operational. But, this claim is not true since for Dienes a number, for example, is more than just something you count with, as will become clear in the sequel.

Perhaps enough has been said to establish the connection with intuitionism and to cast some doubt on the supposed eclecticism of Dienes's theory. In what follows, I will attempt to show that Dienes's fundamental views on mathematics education are consistent with his intuitionist position and that this position is much less eclectic and much more consistent than Bart and Bruner suppose. Thus, Dienes's theory may be classified as an intuitionist theory of mathematics education. I will also briefly indicate how far Dienes went beyond this basic intuitionist theory towards what has become known as constructivism in mathematics education circles.[5] First, however, I will sketch in some of the background by considering Kant's philosophy of mathematics and, in turn, intuitionism.

Kant on Mathematics

Immanuel Kant made two crosscutting distinctions about statements (or judgments). Thus, *a priori* statements are those that reflect knowledge that is independent of experience, while *a posteriori* statements reflect knowledge that is dependent on experience (B2).[6] Necessity and universality are severally necessary and sufficient

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[4] See Dienes (1959b) and Dienes (1960).

[5] For more on intuitionism as a theory of mathematics education and its relation to constructivism, see Fossa (1998).

[6] References of the type (An) indicate the nth page of the first German edition of Kant's *Kritik der reinen Vernunft*, published originally in 1781, while those of the type (Bn) indicate the nth page of the second German edition,

conditions for a statement to be *a priori*. Moreover, a statement is analytic if the concept of the predicate is contained in the concept of the subject. Otherwise it is synthetic (B10).

Of the four possible combinations that these distinctions give rise to, that of analytic *a posteriori* is unrealizable. The synthetic *a priori* is, according to Kant, found in metaphysics, theoretical physics and, most importantly, mathematics. true mathematical judgments are always necessary, hence, *a priori*. But Kant also claims that "all mathematical judgments, without exception, are synthetic" (B14) since they do not follow merely from the "principle of contradiction". Using the statement "7+5=12" as an example, Kant explains that

> *The concept of 12 is by no means already thought in merely thinking this union of 7 and 5 ... We have to go outside these concepts, and call in the aid of the intuition which corresponds to one of them, our five fingers, for instance, or, as Segner does in his Arithmetic, five points, adding to the concept of 7, unit by unit, the five given in intuition.* (B15.)

This astonishing result virtually cries out for more explanation and Kant thus claims that the synthetic *a priori* is only possible on the hypothesis that the faculty of knowledge adds something to that which is known. It does so by adding a formal content to the material content of sense experience. This is done either discursively, which relates to the general concept of relations of things (for example, causality) or through the pure intuitions of space and time (B39-49). The latter are the pure forms of sensibility, the form of all appearances of outer and inner sense respectively. Geometry is given by the pure intuition of space and arithmetic by the pure intuition of time.[7]

Hence, for Kant, mathematics is more than merely definitions and postulates. It is also construction. In Stephan Körner's summary,

> *Kant will not allow that a full description of the structure of space and time requires mere contemplation. It presupposes the activity of construction. To "construct a concept" is to go beyond proposing or recording its definition; it is to provide it with an a priori object ... It does not mean* postulating *objects for it.* (Körner, 1962, p. 28.)

Before leaving Kant, it may be well to note that, on his view, one of the general goals of education is the establishment of a peaceful international community (See Kant, 1960.)

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published in 1787. The second edition contains many substantial modifications. The translations are by Kemp Smith (Kant, 1958.)

[7] Hence, the term "intuitionism" is not the misnomer it is often thought to be.

Intuitionism

The advent of non-Euclidean geometries was widely perceived to knoll the demise of the Kantian view of mathematics.[8] By this time it had become evident, however, that both Euclidean and non-Euclidean geometry can be arithmeticized via analytic geometry. Thus, following Kronecker's admonition to return to our basic intuition of the natural numbers,[9] Brouwer abandoned Kant's apriority of space, but reaffirmed the apriority of time. As he himself put it in his "Intuitionism and Formalism",[10]

> *This neo-intuitionism considers the falling apart of the moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness, may be thought of as a new two-oneness, which process may be repeated indefinitely.* (Brouwer, 1975, p. 127-8.)

Thus, intuitionist mathematics, like Kantian mathematics is about "intuited non-perceptual objects and constructions which are introspectively self-evident" (Körner, 1962, p. 120).

His insistence on the basal nature of mental constructions which are grasped immediately[11] by the mind led Brouwer to reject the principle of *tertium non datur*. In order to ascertain mathematical truth, a constructive proof is necessary. Another intuitionist, Herman Weyl (1949, p. 55), would thus characterize the formalist Hilbert as a mere seeker of consistency, as opposed to Brouwer, a seeker of knowledge.

Still another aspect of Brouwer's constructivism[12] should be mentioned here:

> *Mathematics is a* free *creation: it is not a matter of mentally reconstructing, or grasping the truth about mathematical objects existing independently of us.* (Troelstra, 1988, p. 4.)

Thus Brouwer remains true to Kantian idealism in this regard, although many later intuitionists, the so called "objective intuitionists", do not follow Brouwer on this point.

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[8] Körner argues that this reaction to non-Euclidean geometry is based on a misunderstanding of the critical philosophy. The question, however, is not germane to our argument.

[9] Kronecker insured his own everlasting fame amongst mathematicians by reportedly uttering his *bon mot* at a meeting in Berlin in 1866: "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk."

[10] Brower here calls himself a "neo-intuitionist" in order to differentiate himself from the French intuitionists, mainly Poincaré, Borel e Lebesgue.

[11] That is, without mediation.

[12] Clearly, intuitionism is a kind of constructivism.

This seems to be true of many, but by no means all, of those mathematicians who have tried to combine intuitionism and formalism in a fruitful way. Moreover, "Brouwer's philosophy centers round the individual and ultimately leads to solipsism" (Freudenthal and Heyting, in Brouwer, 1975, p. xv). Part of Brouwer's solipsistic tendencies results from his doctrine that mathematical constructions are purely mental activities and, thus, essentially languageless. Pure intuition is prior to language, which has merely instrumental use in organizing and communicating those constructions. Once again, most intuitionists do not partake of Brouwer's solipsism, but they do share his views on the role of language in mathematics. Thus, R. L. Goodstein (1951, p. 74) claims that language is secondary even in mathematics learning, for "what we teach is an exchange of *things*." Language is only useful in so far as it helps us to get the learner to make his own constructions.

Dienes on Mathematics

Since we have already seen that Dienes was an intuitionist mathematician, we may already have a fairly good idea about how he views the subject matter of mathematics. Nevertheless, it may be well to delineate his position more precisely at this point.

In *Relations and Functions*, Dienes affirms that mathematics is the study of relations. He then goes on to explain that

> *In mathematics the objects of our thoughts are not real concrete objects. They are ideas or abstractions ... The number 3 has no real concrete existence. It is an abstraction built out of sets of objects which all have the properties of threeness, familiar to all of us ... This is where mathematics differs from the experimental sciences. Mathematics is to do with relating non-existent entities to other non-existent entities.* (Dienes, 1976, p. 1.)

Thus, Dienes is, at least, not a Platonist. But what can be meant by "non-existent entity"? How, moreover, has the property of threeness become so familiar to us?

According to the Kantian thesis, accepted by Brouwer, the natural numbers are constituted from the pure intuition of time, where to construct a concept means, as we have already seen, "to provide it with an *a priori* object" (Körner, 1962, p. 28). Thus, natural numbers are not themselves objects, although they contribute to our knowledge of objects. They result, so to speak, from the structural properties of knowledge itself and we become aware of these properties through our self-conscious knowing activities. Dienes claims that "the essence of mathematics is the formation and application of abstract concepts. It begins with the concept of natural number" (Dienes, 1959a, p. 1). Thus, for Dienes, mathematics is the construction of abstract structures in pure intuition.

Dienes on Abstraction

One of the reasons that Bart gives to establish Dienes's supposed realist tendencies is Dienes's "emphasis on abstraction (i.e., induction of concepts from concrete embodiments)" (Bart, 1970, p. 370). Indeed, there does seem to be an overwhelming abundance of evidence in Dienes's writings to support this view. Thus, for example, Dienes writes

> *By concrete, we mean usually, our immediate contact with the real world. We come into contact with objects and events and we re-act to them ... This is the first stage towards abstraction.* (Dienes, 1971b, p. 337.)

Bearing in mind, however, that Dienes is not writing philosophy and, therefore, cannot be held to account by philosophical standards of precision and consistency,[13] one might hesitate to accept Bart's view once it is pointed out that abstraction is not the private property of realism. In fact, it is also commonly found in quite reputable idealist circles. Hence, an emphasis on abstraction does not characterize realism. The question becomes one of how the abstraction process is understood in each case. There is, moreover, at least *prima facie* evidence to suggest that Dienes's is an idealist understanding of abstraction in that his explanation is very similar to that of Hermann Weyl.[14]

The realist generally understands abstraction in one of two ways. The (mostly) earlier view was that abstraction lays bare the objective structure of existing physical reality to the abstracting subject. More recently, however, the subject has been allowed a more active role by emptying the content[15] of "relation" of any empirical content, thereby allowing the subject virtually unlimited freedom in set formation.[16]

The idealist, at least that of the intuitionist stripe, refuses to go along either one of these tracks. On the one hand, there is no objective structure that can be laid bare to the subject and, hence, it is our experience that is structured, not the underlying "reality". But the structure of our experience is the input of the faculty of knowledge itself. On the other hand, set formation is limited to what can be constructed in the intuition from already given or constructed materials.

Now, Dienes often speaks of abstracting from experience, as opposed to abstracting from objects. More important, however, is his description of the process of abstraction. The following medley of citations from *An Experimental Study of Mathematics Learning* will be instructive:

> *The formation of isomorphisms is the process by which we arrive at our abstractions* (Dienes, 1963, p. 59*) ... This [construction] is literally putting things together to build another structure with some previously specified requirements* (ibid., p. 61*) ... A central problem in*

[13] That is not, of course, to say that "anything goes" in mathematics education. Rather, I am just pointing out that one's purpose and one's presumed audience will naturally affect one's style.
[14] Compare Dienes (1961) and Weyl (1949). Although Dienes focuses o the formation of isomorphisms, while Weyl concentrates on establishing equivalence relations, the underlying processes are identical.
[15] Technically this is done by defining relations as sets of ordered pairs.
[16] Some restrictions, however, are still necessary in order to avoid the paradoxes. See further, Fossa (2001, Chapter 2).

> *successfully accomplishing the abstraction process is inducing subjects to form isomorphisms, this being the positive facet of the process* (ibid., p. 85) ... *Constructive thinking takes place when one aims at a set of requirements and attempts to build a structure which will meet them* (ibid., p. 95) ... *Abstraction is essentially constructive in character* (ibid., p.95).

We may conclude, then, that Dienes's emphasis on abstraction does not introduce the taint of realist tendencies into his thought. very much to the contrary, his treatment of abstraction is completely intuitionist in character.

A Psychodynamic Corollary

One of the apparently more puzzling aspects of Dienes's theory is that, according to the Dynamic Principle, "there is a definite process of psychodynamics according to which concept-formation proceeds" (Dienes, 1960, p. 120). At first sight, this tenet would seem more appropriate to the early realists discussed in the last section than to Dienes. But for Dienes, as for Kant and Brouwer before him, all mathematics originates in experience, since, as Kant put it, "objects are *given* to us by means of sensibility, and it alone yields us intuitions" (B34). But our sense experience is structured by the knowing subject and, hence,

> *Experiences have their own structures, and so some experiences will tend to lead towards a concept more quickly than others.* (Dienes, 1959b, p. 16.)

Were Dienes a latter-day realist, it would be virtually impossible for him to hold this view due to their relative liberality towards set formation. By way of contrast, it is a natural corollary to his intuitionism.

The Sixfold Way

Dienes originally proposed a scheme of concept formation involving the three stages of initial play, gradual ordering and final insight. In his later writings (see, for example, Dienes, 1971a or 1971b), however, he expanded this scheme into the following sixfold way to mathematical enlightenment:

Stage 1. Interaction – initial playful interaction with the environment, in which certain regularities are discovered.
Stage 2. Rule Construction – the play becomes structured according to rules.
Stage 3. Isomorphisms – by comparing games with the same structure, this structure is isolated and comprehended.
Stage 4. Representation – the isomorphic situations are represented in an all-embracing form.
Stage 5. Symbolization – the representation itself becomes an object of study.

Stage 6. Formalization – the representation is reduced to a system of axioms.

The scheme is carefully structured to lead the learner from his/her initial experience to abstract axiomatics through a sequence of constructive activities.[17] Such a sequence of activities would probably be the heart and soul of any intuitionist theory of mathematics education. Dienes crystallizes this viewpoint in two principles, the Dynamic Principle and the Constructivity Principle, which are a constant feature in his writings on mathematics education.

According to the Dynamic Principle,

> *preliminary, structured and practice games must be provided as necessary experiences from which mathematical concepts can eventually be built.* (Dienes, 1960, p. 44.)

Once again, this is the Kantian contention (see B34) that intuitions can only be formed through the good offices of sensibility. Moreover, the Constructivity Principle states that

> *in the structuring of games, construction should always precede analysis, which is almost altogether absent from children's learning until the age of 12.* (Dienes, 1960, p. 44.)

This principle is clearly the basic intuitionist contention applied to mathematics learning. The two subsidiary principles of Mathematical Variability – that all inessential features of the concept's structure should be varied – and Perceptual variability – that various perceptual equivalents of the concept to be learned should be presented to the learner – are then presented in order "to induce children to gather the mathematical essence of an abstraction" (Dienes, 1960, p. 44).

At first glance, it might seem strange that an intuitionist would put axiomatics at the acme of mathematics. But, of course, intuitionist mathematicians do use the axiomatic method, which, according to Michael Beeson (1985, p. 82-83), has three advantages: conceptual clarity, rigor and generality. Thus, axiomatics in and of itself does not raise special problems for the intuitionist. What is important for the intuitionist is how one gets to the axiomatics and, as we have seen, Dienes is very careful about this.

Dienes on Language

As would be expected from a mathematics educator, Dienes eschews Brouwer's solipsism, although he does affirm that

> *to teach a concept is essentially impossible. It is only possible to* learn *a concept.* (Reys and Post, 1973, p. ix.)

[17] Note that the last two stages are to be reached only after years of mathematical studies.

This is because for Dienes, as for Brouwer before him, language is not an essential part of mathematics:

> *Mathematics will be regarded rather as a structure of relationships, the formal symbolism being merely a way of communicating parts of the structure from one person to another.* (Dienes, 1960, p. 31.)

Hence, mathematics learning is primarily the building up of structural relationships and only secondarily the apprehension of concept symbolization. In Dienes's view, the latter arises from our need to codify our concepts as an aid to memory and in order to develop methods for further inquiry. In fact, that language is useful in the communication of mathematical constructions is derivative of its original codifying function (see Dienes, 1963, p.152), which in turn depends on experience:

> *Symbolism's power as a tool probably comes from the experience (mental or physical) from which it was originally derived. To step up the supply of power, it is no doubt necessary to go on providing experiences.* (Dienes, 1963, p. 151.)

The construction of *a priori* objects in the intuition is a languageless activity and, thus, "to provide the symbolism before the structure is genuinely learnt is to invite trouble" (Dienes, 1964b, p. 142).

Dienes on Creativity

Brouwer's doctrine on the creative subject is notoriously troublesome and, hence, by fiat, will not be allowed to trouble us here. I will, however, try to explain some of Dienes's views on creativity since they fit nicely into his intuitionist orientation. In fact, Dienes thinks that children are naturally creative and his mathematical games are designed to capitalize on this fact. Thus, "each game was devised to develop some aspect of a child's inborn potential to think creatively" (Holt and Dienes, 1973, p. 13).

Creative thinking in mathematics, however, was soon found to have an unexpected consequence, as Dienes and Golding relate:

> *Every time that a modern mathematical learning situation is established on the basis described in this volume, teachers remark that there has been an almost immediate effect on the artistic expression of the children. This has been such a constant reaction on the part of children taking part in this kind of mathematical learning, it seems that there must be some connection between how we express ourselves in artistic ways and how we express ourselves in mathematical ways.* (Dienes and Golding, 1971, p. 41).

The connection is explained by references to such facts as that both the artist and the mathematician are interested in symmetry.

The superficiality of the foregoing explanation, however, is patent. A more satisfying explanation would result from Dienes's distinction between analytic and constructive thought. Since analysis is always analysis of something, construction must precede analysis. Thus, those who habitually think analytically must approach any given situation by means of preconceived constructions.

> *That is why a constructive thinker is usually a more creative and a more original thinker; his thinking pays off in new and unusual situations, whereas the strategy of the analytical thinker only pays off in the more usual situations from which standard constructions are available.* (Dienes, 1963, p. 116.)

Extrapolating from this analysis, we may conclude that all creativity depends on constructive thought, which is not only an inborn potentiality of the child, but his/her normal way of thought, according to Dienes. If this natural creativity of the child is stimulated in his/her mathematical activities, constructive thought patterns will be reinforced and will likely be applied in other contexts, such as artistic activities, where the child is permitted the necessary freedom for self-expression and self-development.

A Political Coda

In many of his writings, Dienes condemns the authoritarian teacher who equates order with progress and who sacrifices the child's freedom – at times, even her/his dignity – to the almighty syllabus. Thus, he affirms that

> *the essence of a creative learning situation is keenness to inquire, and authoritarianism does not foster a spirit of inquiry.* (Dienes, 1960, p. 46.)

This is, of course, sound pedagogy. If Dienes sometimes seems to go beyond this position, advocating that the mathematics classroom is the appropriate place to instill democratic values in our children, it is not merely the result of his personal reaction to the traumatic events that, during his lifetime, have occurred in his homeland. Rather, it is also another point of contact with Kantian idealism, for, as we have already seen, the establishment of a peaceful international community is, for Kant, one of the fundamental ends of education.

Constructivism

As we have seen in the preceding pages, Dienes's views on mathematics education are in remarkable conformity with Kantian idealism and, more specifically, with an intuitionist account of mathematics. The original purpose of intuitionism, however, was to establish the certainty of mathematics. Thus, when the basic intuitionist tenet, that knowledge is constructed by the subject, was appropriated by various mathematics educators, it was naturally expanded and complemented for the purposes of mathematics education.

The result of this evolution of the intuitionist position in mathematics education is constructivism, which goes beyond intuitionism by stipulating that knowledge is an adaptive process, organizing the experiential world (see Kilpatrick, 1987). According to this "radical" constructivism, not only is certainty lost, but also "the theories, concepts and constructs are culturally and temporally relative" (Lerman, 1989, p. 217).

Dienes certainly has much in common with the constructivists and, in fact, specifically states that all learning is a process of the organism's adaptation to its environment.[18] Nevertheless, Dienes, at least in the period considered in this article, did not abandon the intuitionist quest for certainty. Thus, for example, he affirms that, for any construction to be successful, "the classes that are used for the construction must be well formed" (Dienes, 1963, p. 2). One of the most intriguing aspects of human behavior, however, is that we handle "fuzzy" sets quite capably. That Dienes does not countenance this capability is probably a reflection of his presupposition that mathematics is synthetic *a priori* knowledge. In fact, he seems closer to the original intuitionist position than to that of most present day constructivists, who have come under his influence. Hence, it seems most fair to regard Dienes as a forerunner of (radical) constructivism.

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[18] "Tout apprentissage est un processus d'adaption de l'organisme à son environnement" (Dienes, 1971a, p. 12).

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