

**A BRIEF NOTE ON HOW EINSTEIN'S GENERAL RELATIVITY HAS INFLUENCED THE  
DEVELOPMENT OF MODERN DIFFERENTIAL GEOMETRY**

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**Resumo**

Discutimos alguns aspectos do desenvolvimento da geometria diferencial que podem ser considerados como tendo sido influenciados pela teoria da Relatividade Geral. Mostramos como a busca de Einstein por uma completa geometrização da matéria e do eletromagnetismo deu origem a uma quantidade enorme de trabalhos, tanto em Física Teórica como em Matemática. Salientamos também o papel desempenhado por investigações recentes na Física Teórica na obtenção de novos resultados em algumas áreas da geometria diferencial moderna.

**Palavras-Chave:** Relatividade Geral, Geometria Diferencial, Geometria Lorentziana, Teoremas de Imersão.

**Abstract**

We briefly review a few aspects of the development of differential geometry which may be considered as being influenced by Einstein's general relativity. We focus on how Einsteins's quest for a complete geometrization of matter and electromagnetism gave rise to an enormous amount of theoretical work both on physics and mathematics. In connection with this we also bring to light how recent investigation on theoretical physics has led to new results on some branches of modern differential geometry.

**Keywords:** General Relativity, Differential Geometry, Lorentzian Geometry, Embedding theorems.

## Introduction

It is almost impossible to give a fair account of all consequences brought about by Einstein's scientific work on the development of modern human thought. In physics, Einstein's thought was so revolutionary that practically no branch of this science has escaped its influence. Besides physics, the newborn concepts of space, time and space-time, as well as the quantum nature of the microscopic world, have had great impact also on the field of philosophy. Indeed, as early as in 1922, the French philosopher Henri Bergson [1], who was originally trained in mathematics, wrote a polemical and critical work on the notion of time coming from special relativity. After the formulation of the general theory of relativity and the birth of relativistic cosmology many other philosophers felt immediately impelled to discuss philosophical aspects of the new theory. The British philosopher Bertrand Russell is a good example: his book on the theory of relativity, published in 1925 [2] is a nice and a pedagogical account of Einstein's ideas written with a logical positivism flavour. The apparent success of non-Euclidean geometry to describe our physical world seemed to radically discard the well-established Kant [3] concepts of space and time as a priori notions and this issue is still a subject of debate among philosophers of science. Then comes mathematics. The impact Einstein's general relativity has had in mathematics is immense. No wonder that it has not, as far as we know, been fully assessed by historians of science. Of course we have no intention to embark on such an endeavour here. In this article, our aim merely consists in pointing out a few particular mathematical developments which in our view were directly stimulated by Einstein's ideas. In the first section we give a short outline of the general theory of relativity, some important historical facts and later developments. In the second section, we take as a case study the discovery of some embedding theorems of differential geometry and show how they were physically motivated in the light of modern theoretical physical research.

### The general theory of relativity ( a very brief outline)

Historically the general theory of relativity (1915) grew out of the special theory (1905). The mathematical structure of the later in its original formulation was very simple. However, soon after the appearance of the special theory, two mathematicians, Hermann Minkowski [4] and his contemporary colleague Henri Poincaré [5] made significant contributions to its mathematical structure by realizing that the set of all Lorentz transformations, i.e. those which relate two different inertial reference frames, constitutes a group and that this group leaves invariant a certain quadratic form defined in a four-dimensional space  $M_4$  (or Minkowski space-time). This invariant is now referred in all relativity textbooks as the interval (pseudo-distance) between two events (points in  $M_4$ ). In view of this discovery it would be more natural in the context of special relativity theory to treat space and time no more as separate entities, but as mixed together into a new entity, the space-time. The reaches of this apparently innocuous finding were to be tremendous. Two comments are in order. First, a new branch of mathematics was born. Lorentz invariance stimulated the investigation of a new kind of manifolds endowed with indefinite metrics, now known as semi-Riemannian (or pseudo-Riemannian) manifolds [6]. Second, from the standpoint of physics, there was the hint that the new relativistic theory of gravitation ought to be formulated in a four-dimensional space-time, and that, combined

with the Principle of Equivalence, ultimately led to the geometrization of the gravitation field, and it is here that lies the astounding beauty of the general relativity theory. Physics and geometry are identified and geometrical curvature mimics the effects of gravitational forces acting on particles.

General relativity assumes that in the presence of gravitation our space-time is best represented by a four-dimensional manifold endowed with a Lorentzian metric. It says nothing about the space-time global topology, so in this respect it is still a local theory [7]. An elegant set of partial hyperbolic non-linear equations, found by Einstein and Hilbert [8,9], is used to determine the metric fields from the distribution of matter in space-time. Einstein himself did not expect to solve his field equations exactly and the first solution was obtained by Karl Schwarzschild in 1916 [10]. Schwarzschild's solution describes the geometry of the space-time outside a spherically symmetric matter distribution and contained two puzzling features: the existence of an event horizon and a space-time singularity. Both these aspects of Schwarzschild's solution, which is the prototype of a noncharged static blackhole, were to generate a great deal of mathematical work in the following years.

Very soon general relativity theory was applied to cosmology. In 1917 Einstein wrote a paper in which he modifies the field equations to tackle the problem of finding the geometry of the Universe [11]. His cosmological model described a homogeneous, isotropic and static universe whose spatial geometry may be viewed as the geometry of a hypersphere embedded in an Euclidean four-dimensional space. This was a nice example of a finite universe with no boundaries. However, Einstein's universe did not account for the recession motion of galaxies, observed in 1929 by the American astronomer Edwin Hubble. The discovery of this effect, which was interpreted by the Belgian physicist Georges Lemaître [12] as an evidence of the expansion of the Universe, would drastically change our view of the Cosmos. Indeed, the only plausible explanation of the fact that galaxies are moving away from us is that the Universe is expanding. Curiously enough, an expanding solution of Einstein's original field equations had already been obtained by a little known Russian scientist, Alexander Friedmann, in 1922. Friedmann's time-dependent solution introduced a revolutionary ingredient in our view of the Universe: the idea that the Cosmos started out with a big bang. In mathematical terms it means that the geometry of Friedmann's model, like Schwarzschild's solution, contains a singularity (space-time is geodesically incomplete [13]). Careful investigation of the nature and mathematical structure of singularities found in solutions of Einstein's field equations ultimately led Roger Penrose [14] and Stephen Hawking [15], in the sixties, to discover the famous singularities theorems which have strongly boosted the study of global aspects of general relativity [16] where methods of differential topology have been extensively employed to investigate the problems [17].

### **General relativity and differential geometry**

One of the most cherished projects of contemporary theoretical physicists is to find a theory capable of unifying the fundamental forces of nature, a theory of everything, as it has been called. Unification, in fact, has been a feature of all great theories of physics. In a certain sense Newton, Maxwell and Einstein, they all succeeded in performing some sort of

unification. Twentieth century physics has recurrently pursued this theme. Now broadly speaking one can mention two different paths followed by theoreticians to arrive at unified field theory. First there are the early attempts of Einstein, Weyl, Cartan, Eddington, Schrödinger and many others, whose task consisted of unifying gravity and electromagnetism [18]. The methodological approach of this group consisted basically in resorting to different kind of non-Riemannian geometries capable of accomodating new geometrical structures with a sufficient number of degrees of freedom to describe the electromagnetic field. In this way different types of geometry have been "created", such as affine geometry (asymmetric connection), Weyl's geometry (where the notion of parallel transport differs from Levi-Civita's notion), etc. It is not easy to track further developments of these geometries motivated by general relativity. Already in 1921 the Dutch mathematician Schouten wrote: "Motivated by relativity theory, differential geometry received a totally novel, simple and satisfying foundation" (quoted in [18]). However, the snag with all these attempts was that they completely ignored quantum mechanics and dealt with unification only in a classical level. Of course, an approach to unification today would necessarily take into account quantum field theory. Now the second approach to unification comes into play. It has to do with the rather old idea that our space-time may have more than four dimensions.

The story starts with the work of the Finnish physicist Gunnar Nordström [19], in 1914. Nordström realised that by postulating the existence of a fifth dimension he was able (in the context of his scalar theory of gravitation) to unify gravity and electromagnetism by embedding space-time into a five-dimensional space. Although the idea was quite original and interesting it seems the paper did not attract much attention due to the fact that his gravitation theory was not accepted at the time. Then, soon after the completion of general relativity, Théodor Kaluza, and later, Oscar Klein, launched again the same idea, now entirely based on Einstein's theory of gravity. In a very creative manner the Kaluza-Klein theory starts from five-dimensional vacuum Einstein's equations and show that, under certain assumptions, they reduce to a four-dimensional system of coupled Einstein-Maxwell equations. The paper was seminal and gave rise to several different theoretical developments exploring the idea of achieving unification from extra dimensionality of space. Indeed, through the old and modern versions of Kaluza-Klein theory [20,21], supergravity [22], superstrings [23,24], and to the more recent braneworld scenario [25,26], induced-matter [27,28] and M-theory [29], there is a strong belief among some physicists that unification might be finally achieved if one accepts that space-time has more than four dimensions.

Amidst all these higher-dimensional theories, one of them, the induced-matter theory (also referred to as space-time-matter theory [27, 28]) has called our attention for it recalls Einstein's belief that matter and radiation (not only the gravitation field) should be viewed as manifestations of pure geometry [30]. Kaluza-Klein theory was a first step in this direction. But it was Paul Wesson [28], from the University of Waterloo, who pursued the matter further. Wesson and collaborators realized that by embedding the ordinary space-time into a five-dimensional vacuum space, it was possible to describe the macroscopic properties of matter in geometrical terms. In a series of interesting papers Wesson and his group showed how to produce standard cosmological models from five-dimensional

vacuum space. It looked like any energy-momentum tensor could be generated by an embedding mechanism. At the time these facts were discovered, there was no guarantee that any energy-momentum could be obtained in this way. Putting it in mathematical terms, Wesson's program would not work always unless one could prove that any solution of Einstein's field equations could be isometrically embedded in five-dimensional Ricci-flat space [31]. As it happens, that was exactly the content of a beautiful and powerful theorem of differential geometry now known as the Campbell-Magaard theorem [32]. Although very little known, the theorem was articulated by the English mathematician John Campbell in 1926 and was given a complete proof only in 1963 by Lorenz Magaard [33]. (At this point may we digress a little bit. Campbell, who died in 1924 [34], was interested in geometrical aspects of Einstein's relativity and his works [35] were published a few years before the classical Janet-Cartan<sup>1</sup> [36] theorem on embeddings was established. Manifolds called Einstein spaces had begun to attract the interest of mathematicians soon after the discovery of Schwarzschild space-time and de-Sitter cosmological models). Now compared to the Janet-Cartan [36] theorem the nice thing about the Campbell-Magaard's result is that the codimension of the embedding space is drastically reduced: one needs only one extra-dimension, and that perfectly fits the requirements of the induced-matter theory. Finally, let us note both theorems refer to local and analytical embeddings (the global version of Janet-Cartan theorem was worked out by John Nash [37], in 1956, and adapted for semi-Riemannian geometry by R. Greene [38], in 1970, while a discussion of global aspects of Campbell-Magaard has recently appeared in the literature [39]).

### **Higher-dimensional space-times and the search for new theorems**

Apart from induced-matter theory, there appeared at the turn of the XX century some other physical models of the Universe, which soon attracted the attention of theoreticians. These models have put forward the idea that ordinary space-time may be viewed as a four-dimensional hypersurface embedded not in a Ricci-flat space, but in a five-dimensional Einstein space (referred to as the bulk) [25, 26]. Spurred by this proposal new research on the geometrical structure of the proposed models started. It was conjectured [40] and later proved that the Campbell-Magaard could be immediately generalized for embedding Einstein spaces [41]. This was the first extension of the Campbell-Magaard theorem and other extensions were to come. More general local isometric embeddings were next investigated, and it was proved that any  $n$ -dimensional semi-Riemannian analytic manifold can be locally embedded in a  $(n+1)$ -dimensional analytic manifold with a non-degenerate Ricci-tensor, which is equal, up to a local analytical diffeomorphism, to the Ricci-tensor of an arbitrary specified space [42]. Further motivation in this direction came from studying embeddings in the context of non-linear sigma models, a theory proposed by J. Schwinger in the fifties to describe strongly interacting massive particles [43]. It was then showed that any  $n$ -dimensional Lorentzian manifold ( $n \geq 3$ ) can be harmonically embedded in a  $(n+1)$ -dimensional semi-Riemannian

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<sup>1</sup> Janet-Cartan theorem originated from a conjecture by Schläffli, in 1873, and states that if the embedding space is flat, then the maximum number of extra dimensions needed to analytically embed a Riemannian manifold is  $d$ , with  $0 \leq d \leq n(n-1)/2$ . The novelty brought by Campbell-Magaard theorem is that the number of extra dimensions falls drastically to  $d=1$  when the embedding manifold is allowed to be Ricci-flat (instead of Riemann-flat).

Ricci-flat manifold [44]. As a final remark on the Campbell-Magard theorem and its application to physics, let us note that its proof is based on the Cauchy-Kowalevskaya theorem. Therefore, some properties of relevance to physics, such as the stability of the embedding, cannot be guaranteed to hold [45]. Nevertheless, the problem of embedding space-time into five-dimensional spaces can be considered in the context of the Cauchy problem in general relativity [46]. Specifically, it has recently been shown that the embedded space-time may arise as a result of physical evolution of proper initial data. This new perspective has some advantages in comparison with the original Campbell-Magard formulation because it allows us, by exploring the hyperbolic character of Einstein field equations, to show that the embedding has stability and domain of dependence (causality) properties [47].

### Conclusion

We would like to conclude by pointing out that all these developments essentially grew out of one great theory: General relativity. Underlying this connection between physics and geometry there is the basic idea that a theory of the gravitational field must be a metric theory. Now there is a vast number of metric theories. Their motivation is twofold: quantization of gravity and its unification with the other physical fields. Some of these theories postulate the existence of extra dimensions of the Universe and these multidimensional theories of space-time have employed a complex and sophisticated mathematical language, imported from modern differential geometry and topology. That strange belief on "the unreasonable effectiveness of mathematics in the natural sciences", as put by the physicist E. Wigner [48], seems to be still alive among contemporary physicists. However, this is a mutual process of interaction between the two sciences. In this paper we have tried to explore the other side of this relationship, and how physical research can be beneficial to the development of mathematics itself, in particular the important role Einstein's general relativity has played in promoting progress of some branches of modern differential geometry.

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- [1] H. Bergson, "Durée et simultanéité", Quadrige/Presses Universitaires de France, 1992
- [2] B. Russel, "The ABC of Relativity", Ch. 15, Routledge, London, 1985
- [3] I. Kant, "Critique of Pure Reason", Prometheus Books, Reprint edition, 1990
- [4] H. Minkowski, Phys. Zeitsch. 9, 762 (1908)
- [5] H. Poincaré, "Oeuvres", vol. 9, 500 (1906)
- [6] O'Neill, "Semi-Riemannian Geometry", Academic Press, San Diego, 1983. See also J. K. Beem and P. E. Ehrlich, "Global Lorentzian Geometry", Marcel Dekker, Inc, 1981
- [7] M. Lachièze-Rey and J. P. Luminet, Phys. Rep. 254, 135 (1995). M. J. Rebouças, "A brief introduction to cosmic topology", in Proceedings of the XI th Brazilian School of Cosmology and Gravitation, 188, Ed. M. Novello and S. E. Perez-Bergliaffa, 2004
- [8] A. Einstein, Preuss. Akad. Wiss. Berlin, Sitzber, 844 (1915)

- [9] D. Hilbert, *Goett. Nachr.*, 395 (1915)
- [10] K. Schwarzschild, *Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.- Phys. Tech.*, 189 (1916)
- [11] A. Einstein, *Preuss. Akad. Wiss. Berlin, Sitzber.*, 142 (1917)
- [12] G. Lemaître, *Annales de la Société scientifique de Bruxelles, série A, t. XLVII*, 29 (1927)
- [13] C. J. S. Clarke, *"The Analysis of Space-Time Singularities"*, Cambridge University Press, 1994
- [14] R. Penrose, *Phys. Rev. Lett.*, 14, 57 (1965)
- [15] S. W. Hawking and G. F. R. Ellis, *"The Large Scale Structure of Space-time"*, Cambridge University Press, London and New York, 1973.
- [16] R. Geroch, *"Space-time Structure from a Global View Point"*, in *"General Relativity and Cosmology"*, Proceedings of the International School in Physics "Enrico Fermi" (ed. R. K. Sachs), Academic Press, New York (1971). P. S. Joshi, *"Global Aspects in Gravitation and Cosmology"*, Clarendon-Oxford, 1993
- [17] R. Penrose, *"Techniques of Differential Topology in Relativity"*, SIAM, 1987
- [18] H. F. M. Goenner, *"On the History of Unified Field Theories"*, *Living Rev. Rel.*, 7, 2 (2004)
- [19] G. Nordström, *Phys. Zeitschr.* 15, 504 (1914). (English translation in [21])
- [20] T. Kaluza, *Sitz. Preuss. Akad. Wiss.* 33, 966 (1921). O. Klein, *Z. Phys.* 37, 895 (1926)
- [21] T. Appelquist, A. Chodos and P. Freund, *"Modern Kaluza-Klein Theories"*, Addison-Wesley, Menlo Park, 1987.
- [22] P. Collins, A. Martin and E. Squires, *"Particle Physics and Cosmology"*, Ch. 13, Wiley, New York, 1989.
- [23] See, for instance, P. West, *"Introduction to Supersymmetry and Supergravity"*, World Scientific, Singapore, 1986.
- [24] M. Green and J. H. Schwarz and E. Witten, *"Superstring theory"*, Cambridge University Press, Cambridge, 1987.
- [25] Randall, L. and Sundrum, R., *Phys. Rev. Lett.* 83, 3370 (1999).
- [26] Randall, L. and Sundrum, R., *Phys. Rev. Lett.* 83, 4690 (1999).
- [27] J. M. Overduin, P. S. Wesson, *Phys. Rep.* 283, 303 (1997)
- [28] P. S. Wesson, *"Space-Time-Matter"*, World Scientific, Singapore, (1999)
- [29] M. J. Duff, *Int. J. Mod. Phys. A*, 11 (1996) 5623 (1996)
- [30] A. Einstein, *"The Meaning of Relativity"*, p. 129, Princeton University Press, Princeton (1956). J. A. Wheeler, *"Einstein's Vision"*, Springer, Berlin (1968). A. Salam, *Rev. Mod. Phys.* 52, 525 (1968)
- [31] C. Romero, R. Tavakol and R. Zalaletdinov, *Gen. Rel. Grav.* 28, 365 (1996). J. E. Lidsey, C. Romero, R. K. Tavakol and S. Rippl, *Class. Quant. Grav.* 14, 865 (1997). S. Seahra, P. Wesson, *Class. Quantum Grav.* 20, 1321 (2003)
- [32] J. E. Campbell, *"A Course of Differential Geometry"* (Oxford: Clarendon, 1926).
- [33] L. Maggaard, *"Zur einbettung riemannscher Raume in Einstein-Raume und konformeueclidische Raume"* (PhD Thesis, Kiel, 1963).
- [34] J. E. Campbell's obituary, in *London Mathematical Society Proceedings*, 23, 1XX (1924)

- [35] J. E. Campbell, London Mathematical Proceedings, 20, 1 (1920); 21, 317 (1922); 22, 92 (1923)
- [36] M. Janet, Ann. Soc. Polon. Math. 5 38 (1926). E. Cartan, Ann. Soc. Polon. Math. 6 1 (1927)
- [37] J. Nash, Ann. Math., 63, 20 (1956)
- [38] R. Greene, Memoirs Amer. Math. Soc. 97, 1 (1970)
- [39] N. I. Katzourakis, math-ph/0407067
- [40] Anderson, E. and Lidsey, J. E., Class. Quantum Grav. 18, 4831 (2001).
- [41] F. Dahia and C. Romero, J. Math. Phys., 43, 11, 5804, (2002).
- [42] F. Dahia and C. Romero, J. Math. Phys., 43, 6, 3097 (2002).
- [43] J. Schwinger, Ann. Phys. 2, 407 (1957)
- [44] S. Chervon, F. Dahia and C. Romero, Phys. Lett. A 326, 171 (2004)
- [45] E. Anderson, gr-qc/0409122
- [46] Y. Choquet-Bruhat and J. W. York, "The Cauchy-Problem", in General Relativity and Gravitation, a one hundred years after the birth of Albert Einstein, vol.I, Ed. A. Held, Plenum Press, New York and London, 1980
- [47] F. Dahia and C. Romero, gr-qc/0503103
- [48] E. Wigner, Comm. Pure Appl. Math., 13, 1 (1960)

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