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HISTORICAL ASPECTS OF THE DISCOVERY OF THE EULER CHARACTERISTIC AND SOME OF ITS DEVELOPMENTS IN MODERN TOPOLOGY*

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Abstract

We begin by describing where and when Euler obtained the famous formula $V + F = E + 2$, which relates the number of vertices, edges and faces of a polyhedron that satisfies certain conditions. A few considerations are made about the relation of this formula with other problems and some difficulties of the original proof given by Euler. Then we move to the end of the 19th and beginning of the 20th century when the Euler characteristic and its generalization were linked to new topics in topology. Finally we present some of the generalizations of Euler characteristic which are used in recent (in the past 50 years) developments of topology.

Key words: Euler characteristic, topology, characteristic classes.

Resumo

Inicialmente descrevemos onde e quando Euler obteve a famosa expressão $V + F = E + 2$, que relaciona o número de vértices, arestas e faces de um poliedro que satisfaz certas relações. Fazemos algumas considerações sobre a relação entre esta fórmula e outros problemas, além de certas dificuldades com a demonstração original de Euler. A seguir, passamos ao fim do século XIX e início do século XX, quando a característica de Euler e sua generalização foram relacionadas a novos tópicos em topologia. Finalmente, apresentamos algumas generalizações da característica de Euler que têm sido utilizadas em desenvolvimentos recentes (últimos 50 anos) da topologia.

Palavras-chave: característica de Euler, topologia, classes características.

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1. Introduction

We can associate to a finite polyhedron an integer number. This number is related to the shape of the space, not to its size, whatever this means. If X is a polyhedron, then we denote this number by $\chi(X)$. This number is called nowadays the Euler characteristic (also Euler-Poincaré characteristic) of the space X. The original formulation of Euler's result was (Euler 1758a, Proposition 4):

> "In any solid enclosed by planes, the sum of the number of solid angles and the number of faces exceeds the number of edges by 2." (*In omni solido hedris planis incluso aggretatum ex numero angulorum solidorum et ex numero hedrarum binario excedit numerum acierum.*)

The proof of this result was given by Euler (1758b).

If V denotes the number of vertices, F the number of faces and E the number of edges of a polyhedron, then as a formula we read $V + F = E + 2$. Euler never writes it as $V - E + F = 2$. Despite the obvious fact that the two formulations are totally equivalent and almost the same, the latter formulation has one quite natural generalization which has motivated the definition of the so-called Euler characteristic of finite polyhedra and of certain spaces.

The proof of the main result of Euler, which appears in his second work on the subject (Euler 1758b), had problems and several famous mathematicians tried to fix it. By the end of the 19th century, H. Poincaré worked out a relation between the Euler characteristic and the Betti number of a space. This work provides as a consequence that the Euler number is a topological invariant associated to certain spaces, i.e., it takes the same values if the spaces are homeomorphic. Therefore one can try to use this invariant to distinguish spaces and indeed it has been used successfully. It is worthy to mention that it is not clear if Euler's main interest was to find formulae in the variables V, E, F that could be used to distinguish between spaces or instead to characterize the spaces which satisfy the formulae valid for regular polyhedra. Some people have been interested in this question and this topic has motivated the interesting article by Peter Hilton and Jean Pedersen (1996) entitled "The Euler characteristic and Pólya's dream."

Given a square matrix one can associate to it a number called the determinant of the matrix. Many results can be stated using only this number associated to the matrix, instead of more complicated data of the matrix. In the same spirit, at least to certain spaces, we associate a number to them, the Euler characteristic. It has been shown along the years that much useful information about the space can be stated in terms of its Euler characteristic.

The reader might wonder about many other formulae that one could define in terms of these variables, V, E and F. Is there another useful formula in terms of vertices, edges, faces etc., besides the one that gives the Euler characteristic? In some sense the integer valued function that gives the Euler characteristic is the only reasonable one, at least if we assume that it should satisfy very reasonable properties. This was established by C. Watts (1962).

Later developments motivated by the numerical invariant associated with a space consist of associating with a space, instead of a number, an element of some Abelian group which is attached to the space. This idea includes the original case by considering the Abelian group of the integers, *Z*. As a new situation the group can be taken to be the cohomology of the space. More details and examples will be given in sections 4 and 5.

The purpose of this work is to give the steps which we believe were relevant to the development of the Euler characteristic and some recent aspects of its development in topology. We will try to provide enough references so that the reader can go deeper in a particular aspect of this work of Euler and related topics.

It was a great challenge to write this article in the history of mathematics, a field in which the author has never done any research before. Being aware of the risk involved, I did decide to give it a try, in the hope of presenting a global picture of this very rich and broad topic, including the more recent aspects.

2. The work of Euler about polyhedra and his formula

Leonhard Euler was born on April 15, 1707 in Basel, Switzerland, and died on September 18, 1783 in Saint Petersburg, Russia. He went to the University of Basel when he was 13 years old and got to know Johann Bernoulli. At the age of 16 (in 1723) he got a first degree and at the age of 19 (in 1726) a higher one. He tried, without success, to obtain his first job at the same University. He applied to a position in the physics department. In November 1726 he accepted a job in St. Petersburg at the Russian Imperial Academy of Sciences, supported by Daniel and Nicholas Bernoulli, sons of Johann Bernoulli. Shortly after, in 1731, he became a professor of physiology. On June 19, 1741, he left Saint Petersburg and went to Berlin to the Prussian Academy of Sciences, invited by the King of Prussia, Frederick the Great, where he stayed from 1741 to 1766. During this period he wrote 380 works. Among them two were about polyhedra. The second of these contains proofs of relevant statements made in the first. They were written in 1750 and 1751, respectively, and sent for publication in the 1752-53 volume of the *Novi commentarii* of the Saint Petersburg Academy, which appeared in print only later, in 1758. During his stay in Berlin he was the tutor of the Princess of Anhalt-Dessau, Frederick's niece, to whom he wrote a set of letters about various subjects in science. Those letters, published as a book in 1768, amount to some hundreds of pages, and are by far the most widely read work of Euler. As a result of conflicts with the king and members of the Academy, especially Voltaire, he accepted an invitation to return to St. Petersburg in 1766. He died there on September 18, 1783.

For the complete reference of the two papers mentioned above see Euler (1758a, 1758b). These two papers were brought to the attention of many people since the time of their publication, perhaps firstly motivated by the beauty of theirs results and then by their applications. It turns out that the proof of the main result (Euler 1758b) was not complete. Very prominent mathematicians, among them Cauchy (1813) and Legendre, did some work trying to find a correct and complete proof. A quite interesting publication which refers to the gaps of the proof provided by Euler is the book by I. Lakatos (1976). Let me point out that the difficulty included the lack of a precise definition of the concepts of face, edge and vertex. See more about this in Grünbaum and Shephard (1994, particularly section 6).

Another historical aspect that is worthy mentioning is the relation between the work of Euler and the work of Descartes. Roughly speaking (see Hilton and Pedersen 1996 for more details) we can consider the sum of the face angles of each vertex. This sum does not exceed 2π and the difference between this sum and 2π is called the angular defect. René Descartes proved that the sum of the angular defect over all vertices of a convex polyhedron is 4π. It turns out that this formulation is equivalent to the formula given by Euler (see Hilton and Pedersen 1996 for a proof). Since Descartes (1596-1650) lived before Euler (1707-1783), it is natural to ask if Euler knew the result of Descartes. To know more about this see Hilton and Pedersen (1996) and Grünbaum and Shephard (1994), wherefrom it seems to me that one can conclude that there is no evidence to support a positive answer to this question.

Let us move to the 19th century and see a little of the route started by Euler towards modern topology.

3. Some early applications and generalizations of Euler's formula

A consequence of Euler's formula is that in contrast with the fact that in the plane there are infinite many regular polygons, in the space there are only a finite number of regular polyhedra.

The route started by Euler with his polyhedral formula was followed by the little known mathematician Simon-Antoine-Jean Lhuilier (1750-1840), who worked for most of his life on problems relating to Euler's formula. Lhuilier published an important work on the subject (Lhuilier and Gergonne 1813). One of the results was a consequence of Euler's formula for graphs. Given a graph (a space formed by a union of points and edges), when can it be embedded in the plane? Using Euler's result Lhuilier solved this problem. He also noticed that Euler's formula was wrong for solids with holes in them. If a solid has g holes then Lhuilier showed that $V - E + F = 2 - 2g$. This was the first known result of a topological invariant. One can verify that this number distinguishes almost completely two closed surfaces. If we consider triangulations of the torus and the Klein bottle, we can compute and verify that they have the same Euler characteristic, which is zero. Nevertheless, it is true that two orientable surfaces are homeomorphic if and only if they have the same Euler characteristic. With one extra element associated to a surface (namely, the orientability), in addition to the Euler characteristic, one can distinguish completely between two closed surfaces. Then it was natural to ask about topological invariance and possible generalizations for higher dimensions.

It is clear that the hypothesis that the polyhedron is convex (stated as "solid enclosed by planes") is not necessary in order to have $V - E + F = 2$. In figure 1, we have two examples. In the first example we consider the union of two tetrahedra glued by their bases such that the projection of one vertex falls inside the triangle and the other falls outside, far away. In the second example, we again consider the polyhedron as a union of two tetrahedra. The vertex of the second tetrahedron is in the interior of the first tetrahedron. Euler's convex polyhedra formula had been generalized to not necessarily convex polyhedra by Jonquières in 1890. This was a great step towards the result stating that the Euler number is a topological invariant.

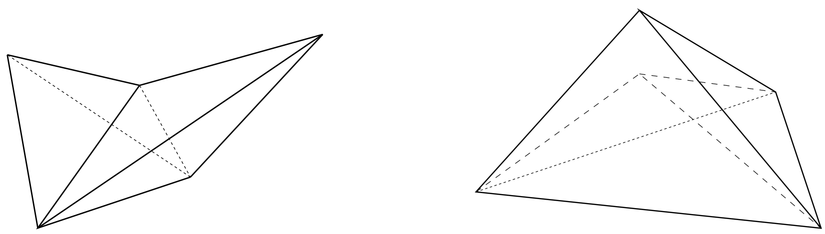


Figure 1. See text for explanation.

Poincaré put Euler's formula into a completely general setting of a *p*-dimensional variety *V* (Poincaré 1899). The idea of connectivity was eventually put on a rigorous basis by him in a series of papers called "Analysis situs". Poincaré introduced the concept of homology and gave a more precise definition of the Betti numbers associated with a space than had Betti himself. Also, while dealing with connectivity Poincaré introduced the fundamental group of a variety and the concept of homotopy.

Before we move on, we may wonder how one could guess the integer valued functions in terms of the vertices, edges, faces etc. which could be useful to distinguish between spaces. So we have in mind such formulae that remain invariant for any possible subdivision of the space in terms of vertices, edges, faces etc. Consider the canonical process to construct a new subdivision from an old one by making the barycentric subdivision. Locally the process consists of making subdivisions of faces. So let us compute the difference between the number of vertices, edges and faces of the old subdivision and the new one for only two polygons. Figure 2 below shows all the numbers. From the data one can read that the sum of the variation of the vertices plus the faces equals the variation of the number of edges. As we make a subdivision of the faces we do not change the shape of the space. This example tells us that the function which maps (V, E, F) to $V - E + F$ satisfies what we want. Furthermore, there is no other linear map on the three variables (V, E, F) which satisfies the condition above besides the multiples of the map $V - E + F$.

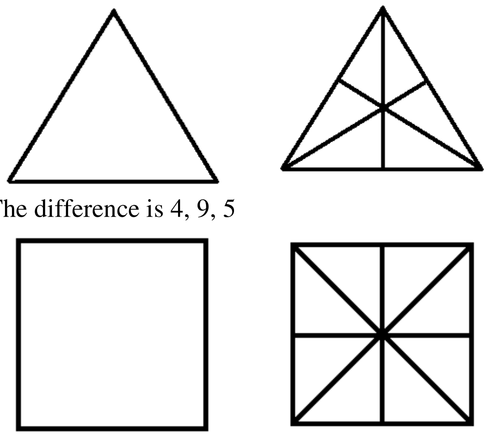
Triangle 3, 3, 1

Subdivision of the Triangle 7, 12, 6

The difference is 4, 9, 5

Square 4, 4, 1

Subdivision of the Square 9, 16, 8



The difference is 5, 12, 7

Figure 2. See text for explanation.

4. Extensions and applications

The starting point for spaces of higher dimension is the sphere S^n, the obvious generalization of the first case studied, the sphere S^2. It was Ludwig Schläfli in 1901 (Schläfli 1901) who computed the Euler characteristic of the sphere S^n. Namely, he shows that $\chi(S^n) = 1 + (-1)^n$, i.e., it is zero if n is odd and 2 if n is even. So, infinitely many different spaces can have the same Euler characteristic. Nevertheless, the next application seems interesting.

Let us consider a differentiable manifold M. Then we have the notion of a tangent space and also of a vector field, which is a family of vectors, one for each point of M distributed in a continuous way. It is an interesting problem to know if there is a vector field such that at every point the vector is not the null vector. In 1926, H. Hopf (Hopf 1926; Alexandroff and Hopf 1935, Satz III, p. 552) proved the following result:

> *There exists a nowhere-vanishing vector field over the manifold M if and only if the Euler characteristic of M, denoted by $\chi(M)$, is zero.*

Observe that the existence of a nowhere vanishing vector field over the manifold M provides an interesting self-map of M. Namely, for each point $x \in M$ consider $\delta\exp_x(v_x)$, where v_x is the vector at the point x, δ is a small positive number and $\exp_x$ is the exponential map at the point x. This is a function which is continuous, is a deformation of the identity map and has no fixed point. There are continuous versions of this result (so M is no longer differentiable), namely one about vector fields and the other about fixed-point-free maps, i.e., maps which do not have fixed points. We explain each one of the cases.

A notion of path fields was introduced by John Nash (1955) and it generalizes the notion of vector field. Then we have the notion of a path field over a topological manifold where we do not have a differentiable structure. A little after, in 1965, R. Brown (1965) proved that a topological manifold admits a non-singular path field if and only if $\chi(M) \neq 0$. Many other results were proved along this line, even in the equivariant context. Very recently, L. D. Borsari, Fernanda Cardona and Peter Wong (2009) have shown the equivariant analog of Brown's theorem for topological manifolds under certain conditions on the action. Their work also provides a clear exposition of the results obtained after Nash, and the notion of the equivariant Euler characteristic. See also the last section of the present paper for the equivariant Euler characteristic.

Suppose that M is a polyhedron, which may even not be a manifold. So we cannot talk about vector fields. Let *id* denote the identity map of M. Certainly *id* has fixed points. In fact, every point of M is a fixed point. The question is to know whether there exists a map g which is a deformation of *id* such that g is fixed-point-free. Again the following result is due to Hopf (Brown 1971):

> *There exists a map g homotopic to the identity which is fixed-point-free if and only if* $\chi(M)$ *is zero*.

The theory of vector bundles became a very important subject for the study of manifolds. One can read properties of the manifold by the knowledge of the bundles over it. The tangent bundle of a manifold is a special case of such vector bundles. Roughly speaking a vector bundle is a collection E of vector spaces R^n indexed by a topological space B, with a topology which locally looks like a product $U \times R^n$. Formally, an n-dimensional vector bundle ξ over a space B is a triple (E, B, p) where p: $E \to M$ is a continuous map such that the pre-image of each point of M is a vector space of dimension n and certain properties hold. For the complete formalization and further properties of these spaces see Husemoller (1974), Steenrod (1951) and Milnor and Stasheff (1974). We can think of a vector field in terms of sections of the vector bundle, i.e., continuous maps s: $B \to E$ such that the composite $p \circ s = id_B$. A vector over a point $b \in M$ is an element of the pre-image of b by the map p. For a given bundle one would like to know when it admits a nowhere-vanishing vector field. Let $E \to M$ be a bundle over M. There are certain invariants associated to this bundle which can be used to derive properties of the bundle and eventually of the manifold. These invariants, on the one hand, do not characterize the bundle but, on the other hand, are more amenable to computation. This phenomenon can occur quite often. One is further to the real object but gains in capacity of computation. The invariants that we refer to are the so-called characteristic classes (Husemoller 1974; Steenrod 1951; Milnor and Stasheff 1974). The characteristic classes are cohomology classes of the base space, i.e. elements in $H^*(M)$, where we purposefully omit the coefficient. Depending on the coefficient various types of characteristic classes are defined. We would like to point out a particular one, which is a cohomology class in $H^n(M, \check{Z})$. Recall that the integer n is the dimension of the vector space of the fiber. This is called the Euler class of the fiber bundle, when one extra hypothesis is assumed, which is orientability

(Husemoller 1974, Definition 7.4, p. 240). What is the relation between this so-called Euler class and vector fields, and why is it called Euler class? For the first question we have: suppose you want to construct a nowhere-vanishing vector field of ξ. A standard procedure to do that is trying to construct such vector field over the vertices, and then extend it over the edges, over the triangles etc. The first possible obstruction to perform this extension is to extend over the higher dimension tetrahedral (simplexes) of dimension n. This obstruction is precisely the Euler class. This means we do have a nowhere-vanishing vector field over the n-skeleton of M if and only if the Euler class is trivial.

For the second question let us consider the special and important vector bundle over a manifold M which is its tangent bundle. In this case if we apply the considerations above, we do have (since the dimension of the base space is the same as the vector space which gives the fiber) that there is a nowhere-vanishing vector field on the manifold M if and only if the Euler class is trivial. But the Euler class is a multiple of the so called fundamental class $\mu_n \in H^n(M, Z) \cong Z$ of the manifold. The factor which multiplies μ_n is the Euler characteristic. Thus, one can recover the result of Hopf. Therefore the Euler class of the fiber bundle can be regarded as a generalization of the Euler characteristic of the base space.

Furthermore, similar considerations can be made for non-orientable fiber bundles. Let us briefly sketch the differences. For simplicity, we will consider only the case of the tangent bundle. In this case we look for a class which lies in $H^n(M, \check{Z})$, where now we have cohomology with local coefficients. This local coefficient system is the system defined by taking the class of a loop in M and associating to it the automorphism of Z which is multiplication by 1 if the loop is orientation-preserving and -1 if the loop is orientation-reversing. If the manifold is orientable, then this is the usual trivial local coefficient system. In any case, for a compact manifold without boundary the result is $H^n(M, \check{Z}) \cong Z$. For a specific choice of a generator μ_M of $H^n(M, \check{Z}) \cong Z$ we have that the Euler class of the tangent bundle of M is the product $\chi(M)\mu_M$. As we can see, the Euler class contains the information about the Euler characteristic of M when we specialize to the tangent bundle. Therefore, the Euler class can be seen as a generalization of the Euler characteristic.

5. Axioms for the Euler number and some extensions

In 1962 Charles Watts published a paper entitled "On the Euler characteristic of polyhedra" (Watts 1962). This work was inspired by the definition of sheaves over a space (Borsari, Cardona and Wong 2009). Let us consider the family of spaces which are homeomorphic to a finite polyhedron. Let Y be a polyhedron and $X \subset Y$ a sub-polyhedron. The main result of the paper is:

Theorem: *Let* ε *be a function defined on the family of the triangulable spaces with base point and which admits a finite triangulation such that*

(1) $\varepsilon(Y) = \varepsilon(X) + \varepsilon(Y/X)$,

(2) $\varepsilon(S^0$ = two points$) = 1$.

Then $\varepsilon(Y) + 1 = \chi(Y)$.

If one forgets axiom 2, the trivial constant map which associates to each space the value zero satisfies axiom 1. This function has no interest. The additive property stated in axiom 1 is quite natural and has motivated further works.

The notion of Euler characteristic goes beyond the spaces homeomorphic to polyhedra. Let (Y, X) be a pair of spaces such that the homology groups with integer coefficients $H_i(Y, X)$ are finitely generated and zero for large i. Then we define the Euler characteristic $\xi(Y, X)$ of the pair (Y, X) to be the integer

$$\xi(Y, X) = \Sigma_{i \geq 0}(-1)^i \, rank(H_i(Y, X)),$$

with the convention that $\xi(Y) = \xi(Y, \emptyset)$. We have the following result:

Proposition (tom Dieck 1979, Proposition 1.6): *If two numbers among* $\xi(Y)$, $\xi(X)$ *and* $\xi(Y, X)$ *are defined, then so is the third, and*

$$\xi(Y) = \xi(X) + \xi(Y, X).$$

This topological Euler characteristic has motivated the following quite general approach. Following Tammo tom Dieck (1979, chapter 5), *an Euler-Poincaré map is a map from a certain category of A-modules to an Abelian group which is additive on certain exact sequences*. This may sound too general but it leads to the interesting notion of the universal Euler characteristic.

The previous example fits in this notion of Euler-Poincaré map by taking as the category of *R*-modules the complexes $\{C_n\}_{n \in N}$ where C_n are finitely generated Abelian groups and C_n is trivial for *n* large, and the map given by the formula

$$\Sigma_{i \geq 0} (-1)^i \, rank(C_i).$$

For the notion of a universal Euler characteristic let us fix a certain category of *A*-modules. Then we consider all maps on Abelian groups which are additive. A pair (e, G) where e is additive and take values on the Abelian group G is called a *universal Euler characteristic for the category of A-modules if given any Euler-Poincaré map* e_1 *with values on an Abelian group H then there is a unique group homomorphism* $\phi: G \rightarrow H$ *such that* $e_1 = \phi \circ e$. A natural problem which has been considered is to find a universal Euler characteristic for a given category of *A*-modules.

In topology we have the notion of *G*-spaces and in particular *G*–CW-complexes, where *G* is a Lie group. For a compact Lie group and a finite *G*–CW-complex one can define a notion of Euler characteristic for such *G*-spaces, also called the equivariant Euler characteristic. As expected, if *G* is the trivial group then we obtain the usual Euler characteristic. One can also define a notion of a universal Euler characteristic for finite *G*–CW-complexes and can be proved that it is unique in some sense. The details are too technical to be described here. All this material can be found in tom Dieck (1979, chapter 5, particularly in section 5.4), where the universal Euler characteristic for finite *G*–CW-complexes is computed.

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