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ON THE PENTAGRAM AS A PYTHAGOREAN EMBLEM

John A. Fossa
UFRN - Brasil

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Resumo

Uma investigação das propriedades matemáticas do pentágono mostra que ele foi usado como um emblema pitagórica para representar a metempsicose. Ainda mais, uma comparação das propriedades matemáticas dos números figurados e figuras geométricas indica que os primeiros pitagóricos sustentaram uma cosmogonia genética, isto é, uma forma rudimentar da teoria de emanações que poderia ter desenvolvido, através de Platão, até o pensameno de Plotino.

Palavras-chave: matemática antiga; matemática grega; matemática pitagórica; pentagrama.

Abstract

An investigation of the mathematical properties of the pentagon reveal that it was used as a Pythagorean emblem to represent metempsychosis. Further, a comparison of the mathematical properties of figurate numbers and geometric figures indicates that the early Pythagoreans had a genetic cosmogony, that is, a rudimentary form of a theory of emanation that may have developed, through Plato, to Plotinus.

Keywords: Ancient Mathematics; Greek Mathematics; Pythagorean Mathematics; Pentagram.

According to Aristotle (*Metaphysics*, A 5), the Pythagoreans thought that number is the first principle of all things and that the world is organized by number-in-harmony, that is, by numerical ratio and proportion. "Number" is to be understood as positive whole numbers. Soon enough, however, these very Pythagoreans discovered that certain line segments are incommensurable, that is, they have no common measure. The discovery, equivalent, in modern terms, to the discovery of irrational numbers, meant, according to the standard interpretation, that there are things in the world that that could not be organized by number-in-harmony. Kurt von Fritz (1945, p. 260) champions this interpretation in no uncertain terms:

> The discovery of incommensurability must have made an enormous impression in Pythagorean circles because it destroyed with one stroke the belief that everything could be expressed in integers, on which the whole Pythagorean philosophy up to then had been based. This impression is clearly reflected in those legends which say that Hippasus was punished by the gods for having made public his terrible discovery.

Hippasus of Metapontum, a very early Pythagorean, supposedly made the discovery in his investigations of the regular pentagon, precipitating a crisis among Pythagorean thinkers that was only to be resolved two or three generations later with Eudoxus' new theory of proportions and that established geometry as the foremost mathematical science for the ancient Greeks. That the discovery purportedly showing Pythagoreanism to be untenable was made in relation to the pentagon was completely ironic because the pentagram (starred pentagon) was an important emblem of the Pythagoreans.

The interpretation rehearsed in the foregoing paragraph is the standard account given by historians and philosophers of mathematics.[1] Nevertheless, some historians have begun to question this account. D. H. Fowler (1987, p. 304), for example, claims that the discovery would not have been seen as a special difficulty in the non-arithmetised and non-axiomatic mathematics of the early Pythagoreans and that the supposed crisis was read into the situation by later writers. In support of this view, Fowler observers that none of the earlier writers, including Aristotle, impute any special impact to the discovery of incommensurables for the Pythagorean program. For somewhat different reasons, in Fossa (2003), I began to move away from this interpretation by characterizing the practical, but not theoretical, ascension of geometry as due to the fact that the Greeks were able to find an axiomatic system for geometry, but not for arithmetic. Herein I will further challenge the standard interpretation by trying to come to grips with the following two questions:

1. What made the pentagram a fitting emblem for the Pythagoreans?
2. What was the relation between figurate numbers and geometrical figures?

Before endeavoring to answer these two questions, however, it would be well to establish a *prima facie* necessity for a new challenge to the standard interpretation. In fact, this can be easily seen, since, if the standard account were correct, we would expect that Pythagoreanism would have been discredited out of hand. Yet we know that this did not happen. Not only did it survive for the two or three generations that it took to resolve the "crisis", it also attracted, during this period, some of the foremost thinkers of the times, including Archytas, Plato and Eudoxus. How could this have happened? Hopefully, by answering the two questions posed above, we will be in a better position to do justice to the historical record.

[1] See, for example, Boyer (1974, p. 53), Eves (1995, p. 106) and Wussing (1979, p. 39).

What was the Pentagram Emblematic of?

Since the major tenet of Pythagoreanism was that all is number and harmony, we may expect that the Pythagorean emblem would reflect this doctrine. Nevertheless, it is hard to see how the pentagram is representative of this tenet. The most striking visual feature of the pentagon, as can be seen in Figure 1, is that its diagonals delineate the pentagram and, therefore, a new pentagon in the interior of the original pentagon.

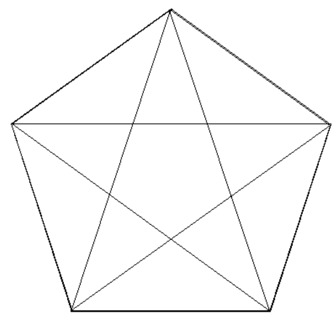


Figure 1.

This self-reproductive property of the pentagon is immediately evocative of another important Pythagorean tenet – that of metempsychosis, the transmigration or reincarnation of the soul – and I propose that it is this doctrine that the pentagram as emblem was meant to represent.

Still, we should ask further "Why the pentagon?" Why not a starred hexagon, for example, or a starred heptagon? These polygons, as do others, also exhibit the same self-reproductive property. Figure 2 shows this for the hexagon and Figure 3 is a heptagon with all its diagonals drawn in.

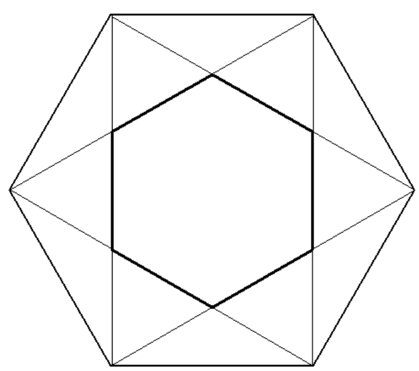


Figure 2.

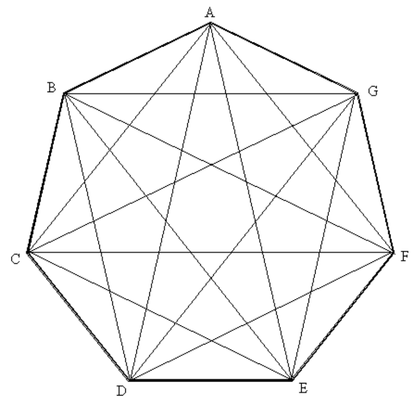


Figure 3.

There are so many diagonals in the heptagon that it is not readily clear what is happening in Figure 3. In fact, however, the heptagon's diagonals reproduce the heptagon twice – once by a starred heptagon with long arms and once by one with short arms (inside a seven pointed star with very long arms). These two cases are separated out in Figures 4 and 5.

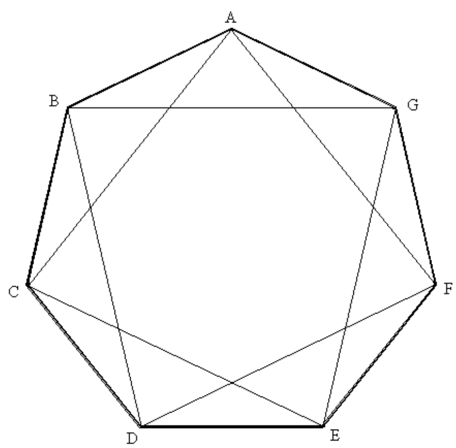


Figure 4.

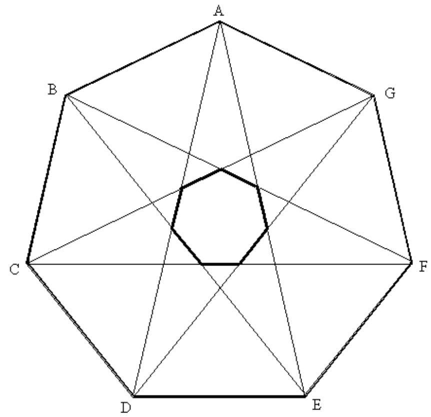


Figure 5.

The redoubled self-reproduction of the heptagon is due to the fact that it's diagonals have two different sizes. When we draw in all the diagonals of the smaller size, that of segment AC, we obtain the long-armed starred heptagon (Figure 4). In contrast, when we draw in all the diagonals of the larger size, that of segment AD, we obtain the short-armed starred heptagon in the seven pointed star (Figure 5). There are also two different sizes of diagonals in the hexagon. The shorter ones produce the starred hexagon, but the longer ones meet in the center of the original hexagon, partitioning it into six equilateral triangles. These results are typical for polygons with an even and odd number of sides. The octagon, for example has three distinct kinds of diagonals, two of which produce starred octagons (with some differences that will not concern us here) and one which partitions the octagon into eight isosceles triangles; the nonagon also has three distinct diagonals, all three of which produce starred nonagons (again with some differences that we may ignore).

The triangle has no diagonals and the diagonals of the square meet in its center, partitioning it into four half-squares. In order to obtain the two squares that these make up, however, it would be necessary to cut and paste these parts. There is, of course, an easy way to decompose the square into four squares by using the perpendicular bisectors of adjacent sides, but that would seem to be a different kind of operation. Thus, the pentagon, with only one kind of diagonal, is the simplest and perhaps visually the most remarkable of the self-reproducing polygons. For this reason, it is likely that it would have been considered the Eminent Mode[2] and, therefore, the most appropriate of these polygons to be used as an emblem.

Surprisingly, there is still another reason why the pentagon would be the most appropriate polygon to be used as emblematic of metempsychosis. This has to do with its relation to the (3, 4, 5) right triangle. According to Roger Herz-Fischler (1998, p. 56-57) the

[2] This terminology was explained in Fossa (1999), where other examples are also given.

Babylonians thought that, when the pentagon is partitioned into five congruent isosceles triangles by lines drawn from the center (more properly, from the center of the circumscribed circle) to the vertexes, these triangles are the double of the (3, 4, 5) right triangle. (See Figure 6.) Actually, this is not a true construction, but only a very good approximation. The Babylonians, however, give no indication of being aware of this fact.

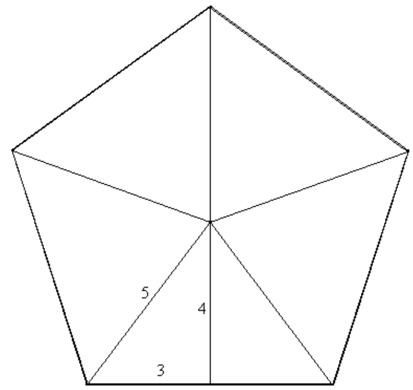


Figure 6.

The Egyptians, it appears, used right triangles to represent procreation since they identified the hypotenuse of the (3, 4, 5) triangle to represent Horus, the son of Osiris, represented by the 4 side, and Isis, represented by the three side. This is entirely appropriate because all right triangles are decomposed into two parts similar to itself by the height to the hypotenuse, as is shown in Figure 7. Later, Plato would use this property of right triangles – again, applied to the (3, 4, 5) right triangle – in his doctrine on the "geometric number", a device that the governors of the Republic were to use to help them determine the correct number of births in each social class and thereby keep harmony in the State. (See Erickson and Fossa, 2001.) In any case, Osiris and Isis were involved in ancient myths relating to the Egyptian belief in reincarnation. Thus, by its association with the (3, 4, 5) right triangle, the pentagon would have inherited a strong connection with metempsychosis. If this interpretation is correct, the combination of the self-reproductive qualities of the pentagon and its association with the sacred (3, 4, 5) right triangle must have made it a very powerful symbol for the Pythagoreans and would explain why it was chosen by them as an emblem of their society.

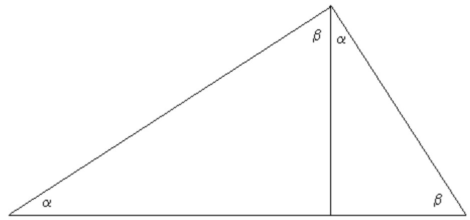


Figure 7.

Admittedly, the evidence tying the (3, 4, 5) triangle to the Egyptian Osiris-Isis myth is given by Plutarch in a context in which he is interpreting this myth in terms of Platonic thought. Nevertheless, as so often seems to be the case with ancient testimony, albeit literally false, it may be metaphorically true. That is to say, stories or interpretations are invented in order to express some deeper truth. Thus, the connection of the (3, 4, 5) triangle with Egyptian metempsychosis myth does point to some kind of Egyptian influence on the Pythagoreans. In any case, the complex of interacting influences from Egyptian, Babylonian and Orphic sources on Greek thought is clearly established by Walter Burkert (2004). Further, B. L. van der Waerden (1983) argues convincingly that the knowledge of Pythagorean triples, as given parametrically, was a prehistoric discovery, transmitted to the Greeks from foreign sources. In fact, the basic triangle of Plato's "nuptial number", as reconstructed in Erickson and Fossa (2001), is not only similar to the (3, 4, 5) triangle, but also seems[3] to be found on the Babylonian tablet Plimpton 322. Hence, there seems to be enough evidence to make the present interpretation viable.

Now that we have answered our first question, it seems rather wrongheaded to suppose that Hipassus discovered incommensurability. It is indeed the very same self-reproductive character of the pentagon, which makes it appropriate as a symbol of metempsychosis, that also reveals incommensurability. This must have been clear to the Pythagoreans when they adopted the pentagram as their emblem. Thus, the persistent, though later, stories about Hipassus reveal what must have been the true historical situation. According to these stories, Hipassus was drowned at sea, not for discovering incommensurability, but for revealing it to the uninitiated. But if the Pythagoreans that adopted the pentagram for their emblem were aware of incommensurability, as now seems to be the case, Hipassus cannot be seen as someone who was punished for revealing a secret that would discredit the Pythagoreans. Rather, he must be seen as being punished for having revealed sacred doctrine to those who were not ready to understand it and thereby profaning sacred knowledge. Indeed, telling the truth, even if it were to embarrass the brotherhood, would not seem to call for *divine* punishment. Profanation of the sacred, however, would be an entirely different matter.

[3] There is some difficulty in interpreting this tablet due to the absence of the zero in the ancient Babylonian number system.

Figurate Numbers and Geometric Figures

The Pythagoreans studied intensively not only figurate numbers, which is consonant with their belief that all is number, but also geometric figures, in which the incommensurable (the *á-logos*, or ir-rational) was present.[4] Thus, it may be suggested, they were not completely successful in trying to combine their scientific and religious doctrines into an integrated whole and ended up with a dualistic philosophy. According to the account being presently entertained, the figurate numbers, based on finite iterations of the unit arranged in certain spatial orderings and obeying arithmetical relationships, would represent their scientific, number theoretic doctrine, whereas the corresponding geometric figures, charged with the mysterious incommensurables, would represent their perhaps inscrutable religious beliefs. The suggestion undoubtedly would be "vindicated" by pointing to Plato, a third generation Pythagorean, whose thought is generally interpreted to be dualistic.

The suggestion presented above, however, runs counter to everything that we've seen in the preceding section. It appears that the religious doctrine is to be found in the mathematics and only in that way does it become Pythagorean. Thus, it would seem that for the Pythagoreans it is always the mathematics that drives the philosophy and, therefore, we would do better to look at the mathematics and see where it leads us. Before going on, we should also make a couple of further remarks regarding the aforementioned suggestion. It is indeed true that the Pythagoreans have been traditionally interpreted as dualists. Their dualism, however, is not centered on a schism between their science and their religion, but on their apparent adherence to dual principles in a purported Pythagorean Table of Opposites. Further, the neo-Platonists were quite insistent on interpreting Plato as already being a neo-Platonist and, thus, not a dualist. Erickson and Fossa (2005, Ch. 7) reviewed these claims and found them to be substantially correct, especially in regard to Plato's fully mature, Pythagorean period.

The Table of Opposites, preserved by Aristotle (*Metaphysics*, A 5), is frankly puzzling in various aspects. The One (*hén*), for example, is listed with the Odd, whereas we know that the One was considered to be both Odd and Even. This may reflect a difference between *hén* and *monás*. This is a question, however, that we need not address here since Aristotle seems to attribute this Table to only one of the various Pythagorean schools or, perhaps, to a related thinker. Thus, we can prescind from the problems posed by this particular Table of Opposites and retain only the principle of opposition.[5]

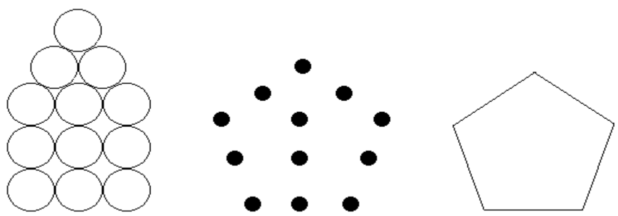
Now, the One or the Unit gives raise to number, for number is just a collection of Units. As the number sequence is examined, however, it immediately falls apart into the Odd and the Even, which reveals the presence of opposition. This opposition, however, is not prior to number, making the Odd and Even into two distinct types. Rather, it is number that is prior and brings about the opposition of the Odd and Even in the different ways that the Unit is iterated. Since "all is number", this basic opposition of Odd and Even will

[4] The symbiotic relations between arithmetic and geometry in Pythagorean thought were already pointed out in Fossa (2003).

[5] For more details on this point, see the discussion in Kirk and Raven (1979, p. 245-247).

manifest itself in different ways as different aspects of reality develop by the articulation of number-in-harmony.

Once that this basic way of the genesis of the world is established, it can be applied to ever more complex situations. In particular, the congregation of number in stable number theoretic relations, represented by the figurate numbers, gives raise to the corresponding geometric figures. The generation of the new geometric figures brings along with them new aspects of reality which stand in opposition to the figurate numbers that bring them about. The figurate numbers, for example, are composed of discrete parts, do not fill up space and manifest *lógos* (the ratio of two positive whole numbers) and *análogos* (proportions among positive whole numbers). The geometric figures, in contrast, are composed of continuous parts, fill up space (or enclose space) and manifest, in addition to ratio and proportion, incommensurable relations. Figure 8 exhibits some of these oppositions in relation to pentagonal numbers and the pentagon.



Two Expressions of a Pentagonal Number and a Regular Pentagon.

Figure 8.

Now that we have appreciated the genetic outlook of the Pythagoreans, we can survey the history of the development of this school in a much more fruitful way than that of the suggestion made in the first paragraph of this section. The sequence early Pythagoreans, Plato, Plotinus can now be seen as a natural line of development from the genetic principles of the early Pythagoreans, through the, at very least implicit, emanation theory of Plato, to Plotinus' full-blown theory of the development of the word as successive emanations from the divine source. Both figurate numbers and geometric figures could be reinterpreted as representing this movement from the One to our multifaceted world. Figure 9, for example, pictures the first few square numbers, each of which can be seen as a new emanation in a cascading sequence. Figure 10 shows the two internal heptagons of Figures 4 and 5 with the star arms suppressed; the result is a Rose of Being, again symbolizing the unfolding of the world in successive emanations.

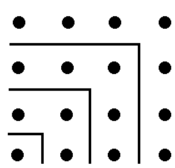


Figure 9.

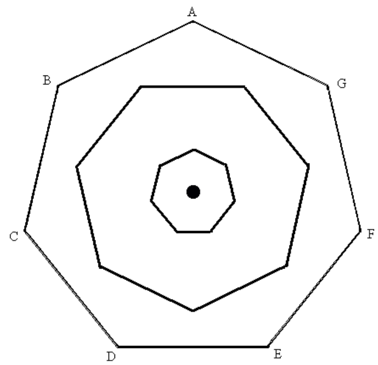


Figure 10.

All this being said, we should also be clear about the following fact: to say that the early Pythagoreans had a genetic interpretation of how incommensurability came into the world and that, hence, this phenomenon was not the trigger for a "crisis" in Pythagorean circles is not to say that the early Pythagoreans were entirely comfortable with incommensurability, and this seems to be indicated by Eudoxus's new theory of proportion and the way that Euclid presents both the new theory and the older one independently in separate books of his *Elements*. In fact, it was only with Euxodus' theory of proportion that incommensurable ratios were made rational by, so to speak, the embedding of them in the whole field of rational ratios.[6]

Conclusion

One of the major problems in interpreting the thought of the Pythagoreans is that the Pythagorean school was not a compact group, well defined in space and time, with a single doctrine common to all its members. Rather, it was a diffuse conglomeration of (sometimes competing) schools, sharing, at times, only the fundamental belief that the meaning (*lógos*) of the world was to be found in mathematics, especially in number-in-harmony. The movement also had a very long history, stretching from Pythagoras to Diophantus. Naturally, different thinkers worked out the consequences of their fundamental mathematical outlook in different ways. The present study of the pentagram as a Pythagorean emblem, however, indicates that there may have been a line of development within the Pythagorean tradition, from the early Pythagoreans, through Plato to Plotinus. This line of development centers on the primality of the One and its articulation of the world, firstly according to genetic principles and, later by a related theory of emanations.[7]

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[6] For more details, see Fossa and Erickson, 2005.
[7] I wish to thank an anonymous reviewer for some helpful comments.

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John A. Fossa
Depto. de Matemática, UFRN
E-mail: jfossa@oi.com.br