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A REFERENCE TEXT OF JOSÉ ANASTÁCIO DA CUNHA: *BALISTIQUE ARITHMETIQUE*, BY PIERRE-LOUIS DE MAUPERTUIS[1]

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Introduction

At the beginning of 2005, Maria do Céu Silva, a researcher at the Oporto University Mathematics Centre, came across six unpublished manuscripts by the Portuguese mathematician José Anastácio da Cunha (1744-1787) in the Braga District Archive. A little later a seventh manuscript was discovered in the same archive by Maria Elfrida Ralha, a University of Minho researcher. This started a process of analysis of these documents by a team of researchers which included the two mentioned above. The aim of this team was to present publicly the results of this initial research at a meeting on José Anastácio da Cunha. This Colloquium took place at the end of 2006, on December 14 and 15 at Minho University, Braga. At the same time, and as a result of a collaboration between Braga District Archive (a part of Minho University), the Mathematics Centre of Minho University, and the Mathematics Centre of Porto University, two books were published. One included the Colloquium Proceedings [6], the other contained the newly discovered da Cunha manuscripts, both in a facsimile version and in transcription.[2].

My task in this process was to analyze two of these da Cunha texts. On one of them I was the sole researcher, but on the other one, the text *La Balistique de Galilée* [1], written in French, the work produced was the joint effort of myself and Professor Carlos Sá, of the Science Faculty of Oporto University. The resulting paper [5] is included in the above-mentioned Proceedings [6].

Da Cunha considered that the written literature on ballistics was lacking a sufficiently simple and quick method which would be useful to gunners, a method which would allow them to train their fire on the targets with relatively good accuracy in a relatively short time. In the historical background described in his paper, only two ballistics works were mentioned, one by Pierre-Louis Moreau de Maupertuis[2] (1698-1759)

[1] I dedicate this paper to Professor Ubiratan d'Ambrósio, my close friend, whose generosity and kindness makes him a unique figure in the scientific community. He has been of paramount importance in stimulating study and research in the history of mathematics in Brazil, and is one of the forerunners and most important contributors in the creation of research links between the Brazilian and Portuguese communities of historians of mathematics.

[2] In the first half of the 18th century, Pierre Louis Moreau de Maupertuis was the main proponent of Newton's ideas in mainland Europe. He wrote on diverse matters, and is considered to be the first to write, in an explicit and structured way, on the transmission of hereditary traits in man. He was elected member of the Paris Academy of

and the other by Edmond Halley (1656-1742). In my paper with Sá we analyze da Cunha's paper, relating and comparing it to those mentioned above.

In this paper my aim is to analyze Maupertuis' short text, called *Balistique Arithmétique* [4], and published in 1731 in the *Mémoires de l'Académie Royale des Sciences*, almost fifty years before da Cunha's paper. From the way da Cunha speaks of Maupertuis's text, we can have some idea of the profound impact it must have had on the scientific community of his time. Almost fifty years after its publication, it was still considered an important reference work [1] (also in [2, p. 106]):

> M. de Maupertuis, auteur célèbre, & excellent Géomètre,[…]débute par annoncer qu'il va renfermer dans une Page plus que ses devanciers n'ont su faire dans leurs grand volumes, & […] Cette briéveté, jointe à la facilité assez élégante de l'Analyse qu'il emploie […] lui ont valu de grands éloges de la part des Connaisseurs. De Savans Géomètres prônent encore aujourd'hui la Balistique Arithmétique […] comme la meilleure qu'il y est au monde.

In fact da Cunha prefers Halley's work, *A Discourse Concerning Gravity, and its Properties, etc* [3]. But as he sees it as also having shortcomings, he decides that his knowledge is worth imparting to the scientific community and its priority target audience, gunnery officers.

The *Balistique Arithmétique*

Maupertuis introduces his work by stating that all that the major treatises have to say on ballistics could fit in a single page, and that this could be done in a more direct and easy way. To do this he will use results from mechanics and from differential calculus.

Suppose a projectile is sent from A towards G describing a curve AMB with an initial velocity $\sqrt{a}$, the velocity in A of a falling body starting from C. Maupertuis takes it as known that the trajectory of the projectile is a parabola:

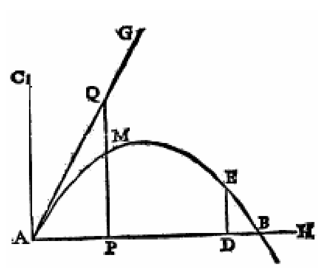

Figure 1

Sciences in 1723, and of the Académie Française in 1743. He became president of the Berlin Academy of Sciences in 1746. There he wrote a short paper ("*Les lois du mouvement et du repos*") in which he formulates the principle of minimum action, which he considered to be his greatest scientific contribution.

Let t=AQ and z=QM, and suppose that when the projectile is at M it has the coordinates (x,y), that is, AP=x and MP=y. Maupertuis states the equality: $\frac{t}{2z}=\frac{\sqrt{a}}{\sqrt{z}}$ holds, that is,

$$t^2 = 4az \qquad (1)$$

From here onwards he designates as n the tangent of the angle of the launch of the projectile, that is, $n = \tan QAP$, and from here he concludes that PQ=nx.[3] Therefore

$$z=QM=PQ-MP=nx-y \qquad (2)$$

From the right triangle [APQ] we have:

$$t^2 = x^2 + (nx)^2 \qquad (3)$$

Substituting (2) and (3) in (1) we get:

$$x^2(1+n^2) = 4anx - 4ay \qquad (4).$$

This equation establishes the relation between the coordinates of each point in the projectile's trajectory, its initial velocity and the inclination of the gun.

Then Maupertuis solves four problems, considering only two variables, the inclination of the gun and the charge of gunpowder:

1. Computation of the angle of the gun for a given projectile with a given powder charge to hit a target E;[4]
2. Computation of the gunpowder required to hit a target for a given inclination of the gun;
3. Computation of the angle of the gun for the projectile to have maximum range;
4. Computation of the minimum charge needed to hit a given target.

In what follows we set out Maupertuis' proofs for each of the four problems quoted. In the first two he uses algebraic methods in equation (4). The last two are solved using differentiation techniques.

1. The target E has coordinates (*b,c*), that is, AD=b and ED=c. Substituting in equation (4) x=b e y=c we have:

$(n^2+1)b^2 == 4nab - 4ac$, that is, a second order equation in *n*. Therefore we obtain:

$$n = \frac{4ab \pm \sqrt{16a^2b^2 - 4b^2(4ac+b^2)}}{2b^2} = \frac{2a}{b} \pm \frac{1}{b}\sqrt{4a^2 - 4ac - b^2}$$

Consequently, if $4a^2 > 4ac + b^2$, there are two possible positions of the gun in order for the projectile to hit the target; there will be only one if $4a^2 = 4ac + b^2$.[5]

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[3] The author takes as the radius defining trigonometric quantities a length R=1

[4] Here he considers the cases where the target is above, on the same plane, or below the plane of the gun.

[5] Maupertuis also considers the hypothesis that the target is at a lower level than that of the gun, a case which is not included in the scheme he presented, indicating only the value of *n*. It is easy to see that the reasoning outlined here can easily be adapted to that case, and the formula thus obtained only differs from this one by a sign, the one of term in *c*, obtaining $n = \frac{2a}{b} \pm \frac{1}{b}\sqrt{4a^2 + 4ac - b^2}$.

2. If the angle is given, n is a constant, and the only variable is a. From (4), replacing x by b and y by c we obtain:

$$b^2(1+n^2) = 4a(nb-c);$$

therefore

$$a = \frac{b^2 + n^2 b^2}{4nb - 4c} \qquad (5)$$

In particular when the projectile hits the horizontal, its coordinates are (b,0). Therefore, for this point we have, from (5):

$$b = \frac{4n}{1+n^2}a. \qquad (6)$$

Maupertuis concludes that the horizontal distance travelled by the projectile is proportional to its force in A, represented by CA.

3. To find the maximum range of the gun with a given powder charge, he considers (6) and differentiates it in order to *n;* he then makes it equal to zero and states that the result is n=1.[6]

4. From (5): $a = \frac{1+n^2}{4nb-4c}b^2$. If we differentiate[7] in order to n and equal to zero, we obtain: $\frac{4[bn^2 - 2cn - b]}{4[bn-c]^2} = 0$, which implies $bn^2 - 2cn - b = 0$, that is, $n = \frac{c \pm \sqrt{c^2 + b^2}}{b}$. As n can only be positive, we get $n = \frac{c + \sqrt{c^2 + b^2}}{b}$.

Consequently, substituting in (5) and simplifying the value obtained, we compute

$$a = \frac{c + \sqrt{b^2 + c^2}}{2}.$$

Final Comments

Maupertuis's way of proceeding is essentially algebraic, in contrast to Halley's, in the paper mentioned by da Cunha [3], and da Cunha's own, which are essentially geometrical. The French scholar's paper seems to have been written with the purpose of showing the advantages of using an alternative way of reasoning to geometry in ballistics, also including the techniques of the new calculus. It is this that Maupertuis has in mind when he says at the very beginning of the paper [4, p. 297]:

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[6] The differentiation leads to $\frac{4a(1-n^2)}{1+n^2} = 0$, that is , $n = \pm 1$. We can only have n=1, as it is a first quadrant angle, therefore $\alpha = \frac{\pi}{4}$.

[7] b and c are constants, as they are the coordinates of E.

> Quoiqu'on ait déja un grand nombre de Traités de Balistique, j'ai crû qu'on ne seroit pas fâché de voir tout cet Art dans une page, qui contient, je l'ose dire, tout ce que contiennent les plus gros Traités, & le contient d'une maniére plus directe, & plus commode pour l'execution, que les constructions géométriques qui dépendent des propriétés du Cercle & de la Parabole.

Curiously enough, da Cunha criticizes him for using such a powerful technique to prove results that can be proven using more elementary methods. Da Cunha asks [1] (also in [2, p. 108]), referring to the third problem mentioned by Maupertuis:

> Y a t'il de l'élégance à recourir aux fluxions pour résoudre un Problême aussi mince que celui de déterminer la plus grande portée ?

But it is precisely this kind of detail that makes *Balistique Arithmétique* interesting, and, in some ways, innovative.

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[6] Various authors, *José Anastácio da Cunha. O Tempo, as Ideias, a Obra…*, Arquivo Distrital de Braga/Universidade do Minho/CMUM/CMUP, Braga: Barbosa & Xavier, 2006.

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