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CULTURE, KNOWLEDGE AND THE SELF: MATHEMATICS AND THE FORMATION OF NEW SOCIAL SENSIBILITIES IN THE RENAISSANCE AND MEDIEVAL ISLAM[1]

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1. Introduction

Some decades ago, to talk about mathematics and culture would not have made too much sense. There was a time –a really long time, indeed– when mathematics was thought of as something beyond cultures. Mathematics, at least in the Western tradition, was thought to deal with eternal objects and the discovery of disembodied truths. Although Platonist views of mathematics have not vanished (Patras, 2001, p. 35), in the past decades, new perspectives on culture, mathematics and mathematical thinking have emerged.

As in all new trends, there are always pioneers who venture to navigate uncharted seas. They go and come back with a chest full of new ideas which, more often than not, those of us who remain on firm land see as extravagances. One of these pioneers was the German philosopher Oswald Spengler who argued that it was vain to see in the number systems of different cultures imperfect versions of something transcendental and unique. In his first book, often written by candlelight towards the end of World War I, the young Spengler claimed that

> There are several number-worlds as there are several Cultures. We find an Indian, an Arabian, a Classical, a Western type of mathematical thought and, corresponding with each, a type of number -each type fundamentally peculiar and unique, an expression of a specific world-feeling, a symbol having a specific validity which is even capable of scientific definition, a principle of ordering the Become which reflects the central essence of one and only one soul, viz., the soul of that particular Culture. Consequently, there are more mathematics than one. (Spengler, 1948, p. 59)

Another pioneer was Raymond Wilder, who, at a time when Bourbaki's structuralist and decontextualized views on mathematics were at their summit, dared to ask questions regarding the relationship between mathematics styles and cultural patterns (Wilder, 1950).

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Ubiratan D'Ambrosio is another of these brave pioneers. Like Wilder, he looked at mathematics from an anthropological viewpoint –a move that was without doubt profoundly innovative. But, in contrast to the eminent American mathematician, D'Ambrosio did not espouse an evolutionary perspective. His intense sensibility to cultural diversity and profound humanistic vision of the world were the result, we can guess, of the subtle merging of his European ancestry with the rich multiethnic environment of Brazil where he grew up. This sensibility to cultural diversity led him to theorize the pluralities of mathematics in a different way. However, from the outset, his endeavor was not motivated by a mere detached and scholarly anthropological interest. Behind what was going to become "ethnomathematics" –one of the most promising research fields at the turn of the millennium– was the conviction that the understanding of other cultures was a fundamental step in reaching peace on earth.

At the core of ethnomathematics rests the question of the relationship between culture, knowledge and the self. The goal of this article is to offer some reflections on this question. We are particularly interested in investigating how, within a certain historical time period, mathematics –in its amplest sense, i.e. as *mathêma* or, in other words, as a reflective cultural manner of knowing (D'Ambrosio, 2006)– accounts for the formation of new social sensibilities –both in terms of capacities to create new forms of understanding and novel forms of subjectivity.

But before going there, in the next section, we sketch a theoretical framework that thematizes the relationship between culture, knowledge and the self along the lines of social praxes. Social praxes, as considered here, involve much more than a simple space for interaction and negotiation of individual wills and interests. Their form of functioning is both afforded and limited by culturally defined forms of social relations (intimately related to ideas of the self and the other) as well as cultural technological forms of production (the sensible dimension of culture through which actions and intensions become materialized). Social praxes are also informed by a supra-symbolic system of beliefs (itself generated by human activity) that shapes –even if only implicitly– cultural understandings of what counts as legitimate –not only in mathematics but also in law, poetry, painting and so on. Questions about the nature of mathematical objects, their mode of existence, the forms of mathematical discourse, etc. find their justification in this supra-symbolic system of beliefs. As we shall see, supra-symbolic systems of beliefs are bearers of historical traditions. But, they are far from static entities. They are kept in motion (and modified) by actual tensions in the always moving space of social praxes. After dealing with these ideas in the next section, we will turn to the problem of mathematics and its social role in the creation of new cultural forms of understanding and novel forms of subjectivity. We will present two case studies. One will be devoted to the Western Late Middle Ages and Renaissance and the other to the Buwayhid period of medieval Islam. We end the article with some general comments about culture, mathematics and the self.

2. Culture and Mathematics

In an article written in 1947 (reprinted in 2006), the American anthropologist Leslie A. White summarized the controversial question about the ontology of mathematics in the following way:

> Do mathematical truths reside in the external world, there to be discovered by man, or are they man-made inventions? Does mathematical reality have an existence and a validity independent of the human species or is it merely a function of the human nervous system?
> Opinion has been and still is divided on this question. (White, 1947/2006, pp. 304-305)

Taking argument against Platonist and Realist ontologies, White challenged the idea that mathematics was about discovering pre-existent truths, as suggested by some of his contemporary mathematicians. In his argument, White had in mind the Platonist position adopted by the British mathematician Godfrey Harold Hardy. Thus, White noted,

> Hardy feels that "in some sense, mathematical truth is part of *objective* reality" […]. But he also distinguishes "mathematical reality" from "physical reality," and insists that "pure geometries are *not* pictures ... [of] the spatio-temporal reality of the physical world." What then is the nature of mathematical reality? […] Mathematics does have objective reality. And this reality, as Hardy insists, is *not* the reality of the physical world. But there is no mystery about it. Its reality is cultural: the sort of reality possessed by a code of etiquette, traffic regulations, the rules of baseball, the English language or rules of grammar. (White, 1947/2006, p. 319)

In a few lines and with great simplicity, White suggested an anthropological answer to the ontological problem of mathematics that had tormented men and women day and night since Plato's time. He was less successful, though, in offering an account of the relationship between cultures and their mathematics. How, for instance, do different mathematics relate to differences in their corresponding cultures? Cast in behavioral terms, his anthropological approach indeed suffers from a questionable cultural determinism in which the individual is seen as a mere reactor to cultural stimuli. Thus, referring to the role of the individual in the cultural growth of mathematics, he said:

> if we wish to explain the discovery as an event in the growth of mathematics we must rule the individual out completely. From this standpoint, the individual did not make the discovery at all. It was something that happened to him. He was merely the place where the lightning struck. (White, 1947/2006, p. 314)

White went on to say that "when the cultural elements are present, the discovery or invention becomes so inevitable that it takes place independently in two or three nervous systems at once." (*op. cit*., p. 315).

White's ideas were not unknown to Soviet psychologists (White visited the former Soviet Union in 1929). However, the Soviets found White's account disquieting. For one thing, it amounts to sinking the individual into some amorphous and faceless collectivity. The problem, the Soviets argued, was too simple a view of culture that remains caught in the meshes of behaviorism. As Aleksei N. Leont'ev –one of the main collaborators of Lev Vygotsky– remarked in a commentary on White's views,

> The world of objects is now replaced by a world of signs and meanings developed by society. Thus, we again stand before the binomial formula, $S \rightarrow R$ [stimulus – response], but now the stimulus is interpreted [not as the stimulus of an object but] as a 'cultural stimulus' (…) human individuals appear as only 'catalytic agents' and 'means of expression' of the cultural process. Nothing more. (Leont'ev, 1978, p. 48).

As interesting as it may be, White's account fails to offer a broad concept of mathematics and culture. While mathematics remains for him "nothing more than a particular kind of primate behavior." (p. 306), culture is reduced to a set of stimuli and a repository of beliefs and codes of etiquette. Moreover, the unrefined relativism of his cultural determinism makes it difficult to see why some cultures develop certain forms of mathematical thinking and not others. Something that is definitely lacking in White's approach is a more nuanced account of the internal dynamics of cultures, i.e. their *social praxes*.

Social praxis as the source of cognition and the self

Social praxis –which in Leont'ev's writings is referred to by the technical term *activity*– is the locus of the emergence of human thought, self and consciousness. It is here that new mathematical objects are created and that, through contact with other people and artifacts, "is realized the process of the individual's acquisition (*Aneignung*) of the spiritual riches accumulated by the human race (*Menschenguttung*) and embodied in an objective, sensible form." (Leont'ev, 1978, p. 19).

There are several dimensions to Leont'ev's idea of social praxis. First, a social praxis is shaped by *cultural forms of social relations, e.g.* forms of interaction between individuals. A simple example is the forms of interaction defined by certain divisions of labour, where individuals assume different roles. Second, a social praxis is shaped by the artifacts that mediate it. This is the technological side of the culture; it involves all the *modes of production* (the material productive forces). Third, from an epistemological point of view, the individuals engaged in a social praxis resort to the *available knowledge* in the culture. Exchanging and loaning, for example, are social praxes which, in their simplest form, involve two parties. In the first case, to produce the exchangeable goods (say shoes for cloth or fish), each party must undergo a process involving other persons (e.g., some family members or "business associates"), cultural knowledge and certain tools. In the second case, to pay back a loan, the party has to produce something equal to what was received (or of equivalent value) through a process that draws on cultural knowledge and involves modes of production and social relations.

But there is yet a fourth element to consider in the study of social praxes. This element is related to a supra-symbolic structure –one that we have termed elsewhere *semiotic system of cultural significations* (Radford, 2006)– which is responsible for the views and beliefs that a culture holds about the good, the right, the beautiful, etc. It includes ideas about truth, the methods to inquire about it, what counts as evidence, the legitimate forms of knowledge representation etc. It also relates to beliefs about mathematical objects, their mode of existence and their relationship to our concrete world. Pythagoreans, for instance, believed that the universe is governed by numbers and that things exist by *imitation* of numbers. Their pebble-based representational technology, used to state and proof theorems (Lefèvre, 1981), clashed with Plato's claim that numbers and mathematical objects in general are unchangeable entities and his view that things in the world exist by *participation* in the Forms –accessible only through *logos* or reasoned discourse (Radford, 2003). All these beliefs reflect accepted principles of an *ontological* nature, i.e. principles about the way the world *is* and the legitimate manner in which it can be scrutinized.

The fourfold view of praxes in terms of (1) forms of production, (2) social relations, (3) available knowledge and (4) their semiotic system of cultural significations is not, however, a mere methodological maneuver. In fact, the crucial point is that cognition and conceptions about the self only arise and develop through the interaction of these four components in which social praxes rest. Figure 1 provides a graphic representation of the four main components of social praxes.

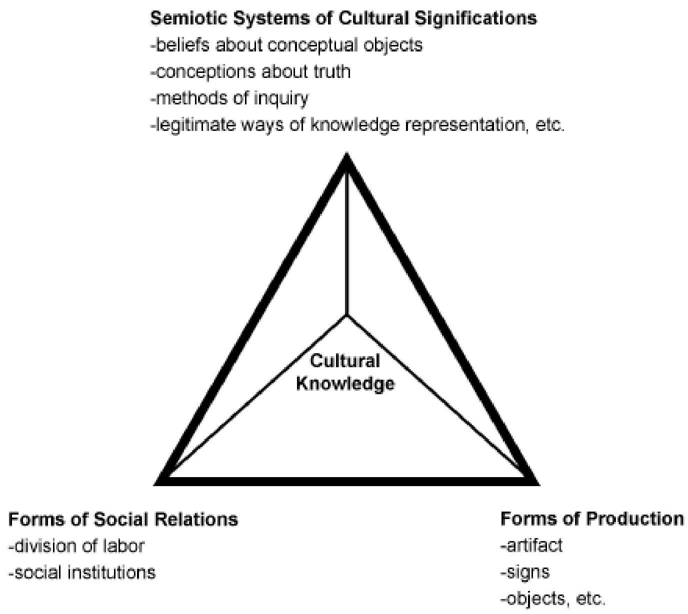

Figure 1. The fourfold structure of social praxis. The segments and planes connecting the vertices suggest the interactive nature between the four key elements of social praxes on the base of a specific historic-economic dimension. The dialectics that result from the mutual connection between vertices leads to particular cultural modes of knowing and subjectivity.

Following the previous threads, in the next section, we will deal with the problem of mathematics and its role in the formation of new forms of understanding and novel forms of subjectivity in the Western Middle Ages and Renaissance.

3. Commerce, Merchants and Ethics in the Western Middle Ages and Renaissance

Anchored at Genoa in the fall of 1163, the vessel of Baldizzone Grasso and Girardo was getting ready for departure. Stabile and Ansaldo Garraton had decided to make a business association and, on September 29th, went to the authorities to write down the terms of their contract. It reads:

> Witnesses: Simone Bucuccio, Ogerio, Peloso, Ribaldo di Sauro and Genoardo Tosca. Stabile and Ansaldo Garraton have formed a "societas" to which, according to their testimonies, Stabile has brought a contribution of 88 liras, and Ansaldo 44 liras. Ansaldo takes the capital, to make it yield a profit, to Tunis or wherever else the vessel that he will take may go—that is, the vessel of Baldizzone Grasso and Girardo. Upon his return, he will hand over the profits to Stabile or to his representative in order to divide them. After deducting the capital, they will divide the profits in half. Done in the Chapter House, September 29, 1163.
>
> Furthermore, Stabile gives Ansaldo the authorization to send this money to Genoa on the ship of his choosing. (Le Goff, 1956, pp. 20-21)

This contract provides us with an important idea of one of the aspects of commercial life in the Western Middle Ages. Here, money is neither merely a good among others –like in non-capitalist exchange contexts (Radford, in press) nor something used to buy goods (as in pre-capitalist societies). Here money is used to *produce* more money.

Behind what at first glance may seem to be a natural action, however, lies a long chain of cultural abstractions. For such an idea to appear, goods first needed to acquire a *use-value*, something not necessarily quantitative –a tenuous and personal mark that indicates the utility of a thing within the individual's daily activities. However, as soon as a good, say A, was exchanged for another good, B, objects acquired a *social* mark–their *exchange value*. The exchange value is not an intrinsic property of the good in question; it is relative to the good against which it is being exchanged. It merely says e.g. how much wax should be traded for a certain amount of wool. Before Stabile and Ansaldo could think of putting their money together "to make more money", people bartered on an everyday basis and from this activity the cultural concept of exchange value emerged. The passage from exchange value to the "proper" value of a good is subsequently a mathematical abstraction. Here, the good is seen as having *its own value*, something which is measured by *money*. There is yet another step in the chain of abstractions: beyond being the *common* quantitative measure of things, money acquired the power to *produce more money*.

In terms of our fourfold view of social praxes (Figure 1), when Stabile and Ansaldo went to see the notary at the Genoese Chapter, they were drawing on an impressive web of historically constituted cultural knowledge. They were also drawing on medieval social and technological structures. First, there was the technological possibility of long-distance maritime transportation offered by the vessel. Second, social institutions –like the Genoese Chapter– presented a social space in which contracts could take place. The contract, as a cultural artifact, was embedded in a rhetoric of *Truth* –e.g. to be valid, contracts required *witnesses*– and the *Right* (the upper vertex of the tetrahedron structure of social praxes; see Figure 1): it tells the parties their obligations; it delineates the frontiers between the right and the wrong and unequivocally establishes the terms according to which the profits will be shared. Contracts, like Stabile and Ansaldo's, portray aspects of the medieval ethics of commerce.

In the Stabile and Ansaldo contract, the mathematics is not very complex. The associates put in amounts of money that are at a ratio of 2 to 1. To account for one of the associates' labour, the profits are not distributed at a 2 to 1 ratio, but equally. Apart from the case where both partners invest the *same* amount of money and labour (and hence share the profits equally), Stabile and Ansaldo's society is, in terms of the mathematics involved, the simplest possible case. We may very well think that Ansaldo could also have put in 40 liras instead of 44. But then the mathematics becomes a little bit more complicated. The amounts of liras written in the contract were not merely coincidental. Had Ansaldo put in 40 liras, "quotidian concepts" in Vygotsky's (1986) sense (e.g. dividing into two) would no longer have been enough. To deal with this case, new "scientific" concepts were required. And this was what actually happened at the end of the Middle Ages and in the Renaissance. The emerging commercial activity led to a refinement and development of mathematical concepts which in turn made it possible to envision more complicated forms of partnership.

For example, the distribution of profits according to the amount contributed by each investor was stated in terms of fractions or numbers that are not in easy ratios, as in the following problem:

> Three merchants have invested their money in a partnership, whom to make the problem clearer I will mention by name. The first was called Piero, the second Polo, and the third Zuanne. Piero put in 112 ducats, Polo 200 ducats, and Zuanne 142 ducats. At the end of a certain period they found that they had gained 563 ducats. Required is to know how much falls to each man so that no one shall be cheated. (Swetz, 1987, p. 138).

This problem does not come from a contract. It comes from the *Treviso Arithmetic* –the first printed arithmetic (1478) which was preceded by a multitude of manuscripts containing similar problems (see, e.g., Arrighi, 1992). The necessity of tackling commercial problems like the previous one in a fair way led to the creation of specialized schools where students got the proper training. These schools began to be set up in the 13^{th} century, the first in Bologna in 1265 (Grendler, 1989, p. 5). Italian city councils had quickly recognized the importance of having a suitable education in the domain of commerce and began to hire teachers (called *maestri d'abbaco*) to instruct children. One document mentions the creation of an *abacus school*, in 1284, in Verona, where Maestro Lotto of Florence had been called to teach (Franci and Toti Rigatelli, 1989, p. 68; see also Høyrup, in press).

Like in the Piero, Polo and Zuanne problem, many of the problems that the maestri d'abbaco composed for the training of students provide us with a window onto gaining a sense of the complexity of the commercial problems that the students would be asked to solve in their professional careers.

In other problems, in addition to putting in a certain amount of cash, the partners entered the company at different times. The following problem comes from a 14th-century manuscript left by a maestro d'abbaco:

> Two men have earned 2768 lire 12 soldi 8 denari. The first entered into the company on the first of January 1390 with 1260 lire, and the second entered on the first of November 1390 with 3128 lire, and when the calends of August 1392 arrived they found they earned, as we said above, 2768 lire 12 soldi 8 denari. What should each receive? (van Egmond, 1976, p. 185).

This problem makes evident the necessity of taking the rhythms of everyday commerce induced by the growing business activity into account. In this context, the *measure of time* became a central concern for the merchants. A precise measure of time was indeed required in order to cope with more complex forms of partnerships, such as those in which the investors, in addition to money, also put in their labour for varying amounts of time. A precise measure of time was also required to deal with the calculation of interest. In one of the problems about interest in the *Liber Abaci* —a problem in which a man placed a certain amount of money at a certain montly interest rate and retrieved a fixed amount of money every year— Leonardo Pisano asks: "It is sought how many years, months, days, and hours he will hold money in the house" (i.e., in the bank) (Pisano, 1202/2002). The religious calendar, based on mobile festivities, was not very effective for keeping track of partnership activities, delayed exchanges, interest, etc. To deal with with these problems, the merchants chose to start their year with the liturgical festivity of the Circumcision –January 1st. The invention of the clock made it possible to think about time in abstract terms (Renouard, 1968, p. 241). It was no longer the individual's actions which organized time to a large extent (e.g. morning prayers, etc.) but the other way around.

However, from the theoretical perspective that we are advocating here, the crucial point is neither to put into evidence what mathematics was produced during this historical period[2] nor to show the underpinning cultural abstractions[3]. The crucial point is rather to elicit *how* mathematics, as a reflection of the world, was instrumental in the formation of new sensibilities. By *sensibility* we mean a subtle progressive cognitive and epistemological change that leads the individuals of a culture to pay attention to themselves and to their world in new ways. Here we are interested in sensibility in terms of capacities to create new forms of understanding and novel forms of subjectivity. In this line of thought, it is important to stress that the mathematics derived from commercial social praxes, such as partnership, mentioned earlier, did not remain confined to the world of merchants or the abacus schools. Progressively, during the late Middle Ages and Renaissance, mathematics penetrated the various spheres of everyday life and offered individuals new modes of action and ways of understanding the world. Of course, not everybody became a professional merchant or a mathematician –far from it[4]. The point is rather that mathematical knowledge

[2] Some of the best accounts of the mathematics of this period can be found in Franci and Toti Rigatelli (1982, 1985, 1988).

[3] For a study about the emergence of algebraic symbolism as a cultural abstraction, see Radford (2006).

[4] The best estimates of the size of the merchant class suggest that, in cities like Venice and Florence, this number was not higher than 10% of the population (Renouard, 1968, p. 236).

came to mediate the relationship between individuals and their culture in more than one respect. The art critic Michael Baxandall (1972) has suggested that there is a commonality between the cognitive skills brought both to partnership or exchange problems and to the making and seeing of pictures. The proportionality of the former and the perspective-based design of the latter are expressions of a same cultural sensibility, the difference being that one addresses commercial and numeric matters while the second addresses visual experience. Because people in urban centres were acquainted, even if only peripherally, with bartering, buying, measuring, selling their labour, etc. in proportional terms, they were sensitive to pictures carrying the marks of similar processes (Baxandall, 1972, p. 101). Just as there was much more use of proportionality and the Rule of Three in commercial transactions in the Quattrocento markets than there had been in earlier times, in the same way, there were, as Baxandall notes, "many more right angles, many more straight lines and many more regular solids in Quattrocento paintings than there are in nature or had been in earlier painting." Figure 2 shows these two common aspects of Late Medieval and Renaissance life.


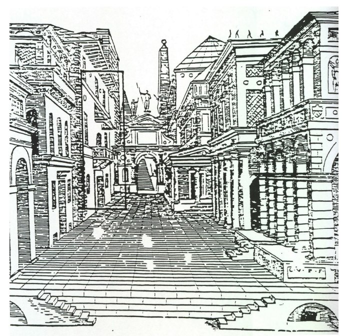

Figure 2. To the left, a drawing from *Trattato d'aritmetica* of Paolo dell'Abacco (14th century) showing a merchant at his table selling goods to a client, while two individuals are engaged in a transaction. To the right, a drawing from the early 15th Century showing the perspective technique for space representation.

Summing up, in the Western Late Medieval and Renaissance periods, mathematics was instrumental in creating new forms of understanding the world. Commercial experience led the merchants to firmly believe that everything has a cause. Through these emerging understandings, they came to realize that precise and complete data is important for foreseeing the future.

> This deep consciousness of the fact that a good piece of information could allow for profitable action through judicious predictions constitutes the logical processes themselves of rational thought. The Italian businessmen of the 14th century behave as though they believed that human reason could understand everything, explain everything and direct all action: they do not express it clearly, but their behaviour shows that they feel it without putting it into words: they have a rationalist mentality. (Renouard, 1968, pp. 227-228)

This new rationalist form of thinking, supported by the increasing sophistication of mathematical methods, is clearly stated in an anonymous 14th-century Florentine manuscript titled "Advices to Merchants", where its author states, "What a mistake to do commerce empirically; commerce is a matter of calculation –si vuole fare per ragione" (quoted in Le Goff, 1968, p.84. Similar views are reported by Bec (1967) in his investigation on commerce and humanism in Florence). It is in this context that merchants and clients, gentlemen and servants were all directly or indirectly living in a world that became dominated by numbers and their proportions and the obsession with the fact that everything –from time, to space, to labour– was measurable (Crosby, 1997; Jeannin, 1957; Swetz, 1987).

The new understandings, promoted by mathematical calculations and the emerging practical rationality, manifested themselves as changes in the "Cultural Knowledge" vertex of the tetrahedron schema of social praxes (Figure 1) and were accompanied, in a dialectical sense, by changes in the other vertices. Thus, new forms of social relations, new technologies and new ideas about the way the world *arose*. Renaissance individuals saw Nature, for instance, in a new light. It was no longer seen as the realm of amorphous, pure matter, the primary source of existence. With Leonardo da Vinci and Galileo, Nature is the realm of order and necessity: it becomes intelligible. The key to opening its secrets rests on the use of proportions. As Cassirer put the matter, in Renaissance thought,

> Proportion may be found not only in numbers and measurements but also in tones, weights, times and places, and in whatever power it happens to be. Through proportion, through the inner measure and harmony, nature is, so to speak, redeemed and ennobled. It no longer stands opposed to man as an inimical or strange power. For, although nature is simply inexhaustible and infinite, we are nevertheless certain that it is the infinity of the *infinite ragioni* of mathematics. And although we can never encompass their entire extension, we nevertheless can grasp their ultimate foundations, their *principles*. (Cassirer, 1963, p. 162)

And it was against this cultural background, erected on the separation of production and commerce and the ensuing division of labourers into producers, merchants, entrepreneurs, etc. that new forms of living and subjectivity, unthinkable in the feudal period, emerged. It is in the unprecedented array of new forms of labour and social relations –which could hardly have been achieved without the generation of the concomitant mathematical knowledge– that one of the chief characteristics of the nascent capitalism also emerged – the new Western concept of the self. As Elias (1991) noted, it was in fact in this historical period that, for the first time, individuals belonged to themselves (as opposed to individuals belonging to the feudal Lord or slaves to their Greek masters[5]).

But practical mental rationality was entangled with a capitalist ethic:

[5] Referring to the Western Middle Ages, Ullman (1966) says: "As a mere subject the individual was no more than a recipient of orders, of commands, of the law, and as a layman, in particular, he was merely a passive spectator who was to obey: his role was that of a learner." (p. 17). Later on, talking about the sphere of the private, Ullman (1966) reminds us that "Whatever they [the medieval subjects] had, that was a matter of royal grace, of royal concession" (p. 19).

> in effect, it was a question after all of businessmen finding the most efficient means of acquiring wealth and making use of that wealth according to a principle of individual pleasure, without knowing any limit to this acquisition or to this pleasure but that of their own satisfaction. (Renouard, 1968, p. 231)

The idea that all profit is good regardless of the manner in which it was acquired (Renouard, op. cit., p. 230), shaped a truly new mode of seeing oneself as well as others. We reach here a form of subjectivity where everything revolves around oneself. The idea of the other is reduced to an instrumental means for the fulfillment of self-interest. Thus, the Renaissance men of business, Renouard complains, "never concern themselves with the living conditions of textile workers and their families" (Renouard, 1968, p. 230).

Naturally, the emerging capitalist ethic clashed with Christian morality, oriented as it was towards the earthly renunciation of all individual ambitions. The Church saw in commercial activity the danger of a desire for profit and a road to the growth of unlimited ambition. With particular vigor, the Church condemned usury, that is to say, every transaction that includes the payment of interest, for *nummus non parit nummos,* "a coin does not make coins". The Old Testament was clear: "Thou shalt not lend upon usury to thy brother; usury of money, usury of victuals, usury of anything that is lent upon usury" (Deuteronomy, XXIII, 19). Or likewise, "Thou shalt not give him thy money upon usury, nor lend him thy victuals for increase." (Leviticus, XXV, 37). The case is clear in the New Testament as well: "And if ye lend to them of whom ye hope to receive, what thank have ye? […] lend, hoping for nothing again" (Luc, VI, 34-35). For the Church, the merchants should not have been looking for profit but merely for compensation for their work. As Saint Thomas argued, it is only under this condition that commercial benefits can be tolerated[6].

But merchants and businessmen were Christians. They did not hesitate to seek God's help in their transactions, which often started with the typical phrase "In the name of God and the Virgin Mary; may they give us and concede to us that our actions become their praise and salvation, to the honour and profit of our soul and our body. Amen."

Likewise, many abacus treatises started with a religious invocation (Høyrup, in press, p. 46). Thus, Docampo Rey, in his interesting work on the training of Catalan Renaissance merchants, quotes the introduction of a Castilian arithmetic manuscript *ca*. 1400 as follows: "In nomine domini amen. Yo el que este libro de arismética que es dicho alguarismo hordene, quiérolo començar en el nombre de aquel que es un solo Dios, Padre e Fijo e Espiritu Santo e estas tres personas es un solo Dios e una natura e una cosa…" (Docampo Rey, 2004, p. 44)[7].

To cope with the contradiction between the self-interested ethic of capitalism and the Christian ethic, businessmen usually gave money to the poor during religious festivities and provided support for churches. The great Minorite church of Santa Croce in Florence was often a beneficiary of their offerings. One of the greatest families of bankers in the 14th

[6] An analysis of usury and the religious environment in the Renaissance is made by doCampo Rey, 2004, pp. 117-122.

[7] "In the name of God, amen. I who have arranged this book of arithmatic, called algorithm, want to begin it in the name of He who is the one and only God, Father and Son and Holy Spirit and these three people are but one God and one nature and one thing..."

century, the Bardi family, had Giotto paint the life of Saint Francis on the walls of the family chapel in this church (see http://www.wga.hu/frames-e.html?/html/g/giotto/s_croce/index.html). Indeed, merchants, businessmen and bankers were all "persuaded that their generosity towards the poor and the churches, as well as their regular attendance at religious ceremonies [was] an assurance against hell" (Renouard, 1968, p. 235).

In some cases, though, ethical contradictions could not be overcome. Some of the most successful merchants, having gone to the limits of the ethics of the nascent capitalism and its concomitant self-centered subjectivity, opted for a return to the ethics of the Church. This was the case of the famous banker Baude Crespin –who finished his life as a monk– or Sebastiano Zani, who retired to the monastery of San Giorgio Maggiore, after bequeathing his estates to the church (for other examples, see Le Goff, 1968).

Let us now turn to our second case study and see how mathematics was intertwined with cultural understandings and forms of subjectivity in the Muslim world.

4. Mathematics, Faith and Law in the Medieval Islamic world

4.1 On the Utility of Mathematics

In his autobiographical treatise, *al-Munqidh min al-□alāl* (The Deliverance from Error), composed towards the end of his life, the renowned Muslim theologian and jurist, Abū □amid al-Ghazālī (d. 1111 CE), explains his decision to abandon a distinguished post as a professor of law in Baghdad and become an ascetic and mystic. His description of his conversion to mysticism and the wonders of inner knowledge in his *Munqidh* was also meant to be a harsh critique of the other intellectual and spiritual paths available to the thinkers of his day: law, theology, Shī'ism and philosophy. Indeed, earlier in his career, al-Ghazālī had exposed the many failings of philosophy and the teachings of the Arab philosophers in his *Tahāfut al-falāsifa* (The Incoherence of the Philosophers). The philosophers, argued al-Ghazālī, with their unfounded confidence in their own sciences and methods of reasoning, overlooked the deeper truths about the world found in revelation (the Qur'ān most notably). The philosophers could not prove their own arguments, they did not agree on anything and were doomed to speak in metaphors. Nevertheless, not all of the philosophical sciences were created equal. While al-Ghazālī wanted desperately to wrest the pursuit of metaphysics away from the revelation-denying philosophers, he did not dispute the utility of such sciences as logic and mathematics. For him, arithmetic and geometry were "matters of demonstration" which would be "impossible to deny once they have been understood and apprehended." (al-Ghazālī, 1982, p. 33) In the *Munqidh*, he further affirms:

> A grievous crime indeed against religion has been committed by the man who imagines that Islam is defended by the denial of the mathematical sciences, seeing that there is nothing in revealed truth opposed to these sciences by way of either negation or affirmation, and nothing in these sciences opposed to the truths of religion. (al-Ghazālī, 1982, pp. 34-35)

Nevertheless, this should not be taken as an unqualified endorsement of the pursuit of the mathematical sciences. For, one could easily fall into the trap of believing that all of the philosophical sciences were as clear and free from conflict with religious truths as mathematics:

> . . . every student of mathematics admires its precision and the clarity of its demonstrations. This leads him to believe in the philosophers and to think that all of their sciences resemble this one in clarity and demonstrative cogency. Further, he has already heard the accounts on everyone's lips of their unbelief, their denial of God's attributes, and their contempt for revealed truth; he becomes an unbeliever merely by accepting them as authorities . . ., and says to himself, 'If religion were true, it would not have escaped the notice of these men since they are so precise in this science.' (al-Ghazālī, 1982, p. 33)

Al-Ghazālī had already reprimanded the philosophers on this account in his *Tahāfut*. It would seem that they knowingly used mathematics and logic as tools for indoctrinating the innocent into their unorthodox beliefs. Thus, the philosophers

> use the appearance of their mathematical and logical sciences as evidential proof for the truth of their metaphysical sciences, using [this] as a gradual enticement for the weak in mind. Had their metaphysical sciences been as perfect in demonstration, free from conjecture, as their mathematical, they would not have disagreed among themselves regarding [the former], just as they have not disagreed in their mathematical sciences. (al-Ghazālī, 1997, p. 4)

We should not view al-Ghazālī's penchant for drawing a sharp line between philosophy on the one hand and revelation on the other, or likewise, between mathematics and the rest of the philosophical sciences (such as ethics, politics and metaphysics) as something exceptional in the history of Islamic thought. By the time the famous theologian was writing and teaching in Baghdad at the turn of the twelfth century CE, a firm dichotomy had already been established between, on the one hand, the pursuit of the so-called "first science" or "science of the Ancients" (*'ilm al-awā'il*), which Islamic civilization had inherited from the Hellenic world which had preceeded it in the Middle East, and, on the other hand, the Islamic sciences (*'ulūm al-islāmīya*) (Makdisi, 1981 p. 79) or sciences of the faith (*'ulūm al-dīn*). The most important of these was the law (*fiqh*) which regulated every aspect of a Muslim's daily life and a great deal of effort was expended in the analysis of its sources, namely, the Qur'ān and the traditions (*hadīth*) regarding the conduct of the Prophet Muhammad and the early Muslims.

This is not to say that the two great bodies of ancient knowledge and religious knowledge were kept completely distinct: the reciprocal influences between Islamic law and theology and "Greek" philosophy can be clearly discerned. The well-rounded medieval Muslim scholar might be versed, at the same time, in the law and traditions of his faith as well as in one or more of the philosophical sciences, which had been translated from Greek, Syriac, Persian or Sanskrit into Arabic and which were being developed by Muslims on a daily basis. The age of al-Ghazālī, however, was one in which the Islamic sciences and the law in particular were benefiting from a growing institutionalization, while the sciences of the

Ancients were receiving less sponsorship from the ruling classes. The Islamic and foreign sciences nevertheless continued to co-exist closely in a city like Baghdad and the ruling elite, along with al-Ghazālī, could not ignore the great utility of a science such as mathematics, which had already contributed so much to the technological advancement of the Muslim empire. In the Islamic world, as we shall try to show in the section which follows on the Buwayhid period in medieval Islam, mathematics could fulfill both the conceptual ideals of the Platonizing philosopher as well as the practical needs demanded by the daily activities of the engineer or the jurist.

4.2 Mathematics and Aesthetics in Al-Karajī's al-Fakhrī

The century or more of Buwayhid rule in Iraq and parts of Iran (945 CE – 1055 CE) represents an important period of transition in medieval Islamic history: the transition from the age of the classical Caliphate (the unified Muslim empire that had been expanding since the death of Muhammad in 632 CE) to the fragmentation—but not destruction—of that same empire into various and mostly autonomous successor states ruled by military overlords who were minimally subservient to the Caliph in Baghdad. The Buwayhid family, military overlords of Iranian extraction, took the additional step of marching on Baghdad and placing the Caliph and the Muslim community under their protection and *de facto* rule. Certain elements of continuity persisted: a well-organized bureaucracy was, as ever, in place and the Caliph remained the spiritual and moral leader of the Muslims. He was the one who ensured that justice would be carried out according to the premises of Islamic law. However, the Buwayhids also introduced one of the most important and long-lasting economic innovations of the medieval period, namely, the *iqṭā‘*, a non-hereditary parcel of land given out to outstanding military officers who could then tax the land for their own financial gain during their lifetimes. An institution similar to—but not to be confused with—the medieval European *fief* and European feudalism, the new *iqṭā‘* system and its consequences for agriculture, taxation and the economy in general would have posed new challenges at all levels of the administration. Although the state in the form of the ruler, in this case, the Buwayhid Emir, was ultimately in control of the land and its distribution, the soldier who had been given the use of it and those cultivating it now entered into a new economic relationship (Lapidus, 1988, pp. 146-152). Furthermore, the Buwayhid period was also one of significant renewal for the religious sciences and for the philosophical sciences in particular. At its height, in the ninth century CE, the classical Caliphate had been the most important sponsor of the translation of scientific texts from Greek and other languages into Arabic as well as a strong advocate for the validity and value of that enterprise. This fervent intellectual activity necessarily subsided as the Caliphate faced the economic and political crises that eventually led to the Buwayhid takeover. However, once in power, the Buwayhids were quick to revive the culture and the excellence in all disciplines that the teeming metropolis of Baghdad was famous for (Kraemer, 1992, pp. 46-60).

It was precisely in 10th-century Baghdad that Abū Bakr Muhammad b. al-Hasan al-Karajī (d. 1019) was born. We know little of his biography. He was a high-ranking official of Iranian origin in the Buwayhid administration. His extant works testify to his mastery of algebra and the other mathematical sciences as well as to his expertise in the building of *qanāt*s or the subterranean tunnels used to irrigate the arid regions of Iran and Iraq (Vernet, 1978). In his *al-Fakhrī fī al-jabr wa-al-muqabāla* (The Book Dedicated to Fakhr al-Mulk on Restoring and Balancing [i.e. Algebra]), al-Karajī sets himself the task of improving upon the art of algebra. In his introduction, he states that he has written his treatise after a long and rather obscure period of tribulation in honour of the Buwayhid vizier Fakhr al-Mulk whom he credits with having restored justice and the good to the administration. Other than this praise directed towards his benefactor, for whom the treatise is also named, the mathematician gives no other hint about his potential readership. In all likelihood, it was directed towards those with an interest in the intricacies of algebra. At the same time, as several of the problems in the treatise suggest, the work would have been of use and of interest to others working within the administration and who might have to solve problems on a daily basis. As al-Karajī argued, the best way to determine unknowns is through the art of algebra thanks to its "power and uniformity in all problems related to calculation" (Woepcke, 1982, p. 45). Inspired by Diophantus' *Arithmetica*, the *Fakhrī* is one of many works of Arabic science to appropriate and then seek to advance the learning of the ancients (Vernet, 1978). One of the innovations is the more concrete or applied context of some of its problems. Indeed, while in his *Arithmetica* Diophantus dealt with problems about numbers, al-Karajī introduced some problems which, to some extent, show an effort to use algebra as a form of understanding his concrete world.

By way of example, we can consider the following problems:

> A servant receives, for a month's (30 days) wages, 40 dirhems and a ring, the ring representing 5 days wages; what is the price (x) of the ring? (Woepcke, 1982, p. 77)
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> If he receives 40 dirhems as a month's wages, a ring and a piece of clothing, which represent 3 days and 6 days wages respectively; what is the price (x) of the ring and what is that (2x) of the piece of clothing? (Woepcke, 1982, p. 77)

These as well as some other problems encountered in the *Fakhrī* do not point to a context particular to the Buwayid period. Some problems are merely characteristic of medieval Muslim society in general[8]. In the two problems cited above, the concrete situation discussed has to do with the monthly salary of a servant, presumably paid out by his master in dirhems and goods. Given the data available about the salaries of the time period and keeping in mind that treatises such as the *Fakhrī* are often used by historians to determine average prices and salaries (Ashtor, 1969, p. 112)[9], we can say that the transactions presented in the two problems above are in keeping with what we know about labour and wages in the medieval Muslim world.

[8] We are grateful to Professor Jeff Oaks for calling these problems to our attention.

[9] Indeed, in the section of his work devoted to prices and salaries during the Buwayhid period, Ashtor makes use of al-Karajī's *al-Kāfī fī al-□isāb* (The Sufficient [Book] on Calculation) to tell us that the average monthly salary for a worker could be between 35 to 47 dirhems, a range that the salary of the servant in the *Fakhī* falls into as well (Ashtor, 1969, p. 112, note 8).

Wage labour was a common phenomenon throughout the Islamic world. The rate of wages was determined by the availability and demand for labourers and goods and was regulated by the juristic authorities (Shatzmiller, 1997, p. 179). It helps to know that around the time of the composition of the *Fakhrī*, the dirhem, a silver coin, was scarce in Baghdad and was being replaced more and more by the gold dinar because dirhems were being diverted towards the countryside for *iqtā‘* related transactions (Cahen, 1952, p. 341). It was not unusual for a worker to be paid partly in money and partly in goods, a practice that was in accordance with the regulations for remuneration set down in the religious law, which, ideally, was supposed to guide all such transactions. The individual's labour was seen as a quantifiable entity, as was the time he or she spent labouring. This seems to have been true since the time of the Prophet and jurists would have used traditions coming from him in order to establish what constituted the just and the good in any financial transaction. Customary practice often influenced actual practice and some schools of law (of which there are four major ones in Sunnī Islam) used juristic prerogative in order to accommodate certain habitual and widespread practices of merchants, craftsmen and tradesmen (Udovitch, 1970, pp. 250-1). Standards of payment would have been generally known and the state had many administrators in place, such as the *muḥtasib*, a market inspector and policeman of public morality, who would go around the city making sure that the economic and ethical norms of the law were being respected (Shatzmiller, 1997, p. 179).

To add to the above, we may consider the question of contracts carried out between individuals that show that the law, depending on the school in question, could accommodate many kinds of exchanges and transactions as far as labour and capital, but not usually goods or merchandise, were concerned. The law of contracts, indeed, had been evolving since the earliest days of Islam, inspired by the practices of the pre-Islamic merchants from the Arabian Peninsula engaged in long-distance trade (Udovitch, 1970, pp. 172-173). Once again, some jurists were willing to accommodate practices that were not entirely in keeping with the fixed regulations set down by the religious law, given the great variety and quantity of such transactions taking place during the medieval period. As one scholar has pointed out, with reference to the work of S.D. Goitein, the renowned historian of medieval Muslim social history, commercial partnership contracts could also be used as labour contracts:

> . . . in medieval Near Eastern society the *commenda* and partnership contracts fulfilled essentially the same function and to a large extent displaced employment and loan contracts. Labour partnerships and industrial *commendas* were quite frequently nothing but veiled forms of employment . . . . it is apparent that employment as a form of economic collaboration was eschewed, since dependence upon others for a livelihood was considered degrading and humiliating. Consequently, many enterprises, no matter on how modest a scale, requiring the combined efforts of more than one person would be organized in the form of partnerships and *commendas* (Udovitch, 1970, pp. 184-185).

Such partnerships and exchanges of labour for some kind of remuneration could take place on a long-distance or local basis. Some schools allowed not only the exchange of capital or equipment for the individual's labour and time but also the exchange of raw materials in exchange for the craftsman's ability to transform them into a sellable commodity whose profits could then be shared by the parties involved (Udovitch, 1970, pp. 185-6). We therefore see the prevalence of a symbiotic relationship between the law and the economic sector. Just as the religious law regulated all aspects of the believer's spiritual life, so too did it serve to organize his daily life in the world of work through its emphasis on the just quantification of labour and its remuneration. As Michael Cook has pointed out, it is important to remember that in Islam "an executive power of the law of God is vested in each and every Muslim" (Cook, 2000, p.9). It was therefore incumbent upon the individual believer to uphold what was right and just and prevent what was wrong. We might add that, just as in Christian Europe, usury and the self-interestedness it entailed was highly frowned upon by the piety minded and forbidden by the religious law. However, although Muslims might be subject to the religious law it also gave them a certain amount of agency and was in place for their protection: even the slave was accorded limited rights to remuneration and marriage (Lapidus, 1988, p. 148). Furthermore, the jurists and other religious functionaries often carried out their duties as individual believers and not necessarily agents of the state. They determined the correct times for and number of daily prayers but also provided an ideal framework in which daily economic transactions could take place. Certainly, like in the Western world, mathematics had a prominent role to play in the cultural sphere.

There are, however, problems in the *Fakhrī* which, while stated in a seemingly real context, are, in fact, expressions of an aesthetic mathematical sensibility. Three of these problems are the following:

> Of two messengers who leave at the same time, the first makes 10 *farsang*s[10] each day, the second in succession 1, 2, 3 *farsang*s, etc. How many days will it take for him to reach the first [messenger]? (Woepcke, 1982, p. 82)
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> If the first make 11 *farsang*s each day, having left 5 days before the second, when will he be reached by the latter? (Woepcke, 1982, p. 82)
>
> Having left on the same day, they go, the first in succession 1, 3, 5 . . . *farsang*s, the other 10 *farsang*s each day. How many days will it take them to meet each other? (Woepcke, 1982, p. 82)

These problems refer to a concrete context, but the messengers would hardly travel as stated in the stories. These problems, we want to suggest, reflect an aesthetic dimension of mathematical thinking, a narrative way of going beyond the concrete, like tales told in paintings. (Of course this aesthetic dimension is not limited to al-Karajī's work. For a discussion of the abacists' aesthetics in the Renaissance see Radford, 2003, p. 135; see also Bento Fernandes' *Tratado da arte de arismetica* (1555) where the aesthetic dimension is conveyed in terms of recreational problems[11]).

A possible context for these problems emerges when we consider them in conjunction with a mathematical treatise by a contemporary of al-Karajī's, Abū al-Wafā' al-Būzajānī (d. 997

[10] A unit of measurement from the Persian "parsang".

[11] This is the case of a problem in which the number of eggs in a basket turns out to be, after calculations, equal to 27,719 (Silva, 2007).

CE).[12] Indeed, al-Būzajānī presents a similar problem about messengers that also provides a context in which the given calculations might be carried out in daily life:

> The *sultān* decrees the sending of a messenger to a distant locality, 100 *farsakh*s[13] away from the court, with orders to make the journey in segments of 12 *farsakh*s per day: how much time will it take? . . . If the *sultān* gets a new idea, and he wants to send a second messenger to catch up with the first the next day or on the night before, or a day before, his arrival in the city . . .(al-Būzajānī quoted in Cahen, 1952, p. 337)[14]

The problem of the messengers, as put forth by al-Būzajānī, furnishes us with a cultural context from which to understand it beyond its role as a mathematical exercise. It tells us something about the practical problems of communication. Other problems, gleaned from al-Karajī's treatise, also reflect the demands placed on the mathematician in service to the state to be able to work out various administrative challenges mathematically. For example, keeping in mind what we have said about the introduction of the *iqtā'*, in the *Fakhrī* we find several problems dealing with the distribution of profits between the cultivators of a piece of land and its master or proprietor:

> Of a quantity of dates, 3/5 go to the proprietor and 2/5 to the farmer; the master receives 7 *jarīb*s 5 *qafiz*[15] more than the farmer. What is the total quantity (x) of dates? (Woepcke, 1982, p.80)
>
> Of 100 *jarīb*s 3/5 go to the proprietor and 2/5 to the farmer. The farmer takes a certain quantity (x) and the proprietor [takes] the rest; then the farmer gives back ¼ of what he has taken to the proprietor and the proprietor gives the farmer 1/5 of what he has taken, after which, each has what was due to him. How much had each taken in the first place? (Woepcke, 1982, p.80)
>
> The same things being supposed, the farmer gives 5 *jarīb*s back to the proprietor, after which each has what is due to him. How much (x) had the farmer taken [initially]? (Woepcke, 1982, p.80)

We might assume that in these cases it is a matter of determining an equitable division of profits between the temporary beneficiary of the *iqtā'* and the peasant cultivating the land for him. Such a relatively small transaction, in effect between individuals, can be contrasted to the more elaborate calculations needed to determine the profits that the state or ruler would have been entitled to through taxation before the parceling out of the land. In the problems cited above, we see the fourfold structure of social praxis (Figure 1)

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[12] This text, entiled, *Kitāb al-□āwī li-al-a'māl al-sultānīya wa rusūm al-□isāb dīwānīya* (The Comprehensive Book on Royal Service and the Regulation of Administrative Calculations) is a manual for bureaucrats and is discussed at length in Cahen (1952). Interestingly enough, Cahen makes use of al-Karajī's mathematical treatise *al-Kāfī fī al-□isāb* to comment on al-Būzajānī's text and to help explain some of the data he found therein. He also notes the likely influence of al-Karajī's work on that of al-Būzajānī.

[13] The same unit of measurement as the "farsang" or "parsang".

[14] Cahen indicates that al-Būzajānī continued the problem by adding a third messenger and then supposing that the messengers did not all leave from the same place (Cahen, 1952, p.337, note 23).

[15] The *jarīb* and *qafiz* are units of measure for grains and other dry goods.

coming into play: social relations and forms of production were brought to bear on cultural forms of knowing and using mathematics.

4.3 Mathematics and the Law

We must now ask what relationship the religious law, as the ultimate regulator of daily life and the undisputed "queen of the Islamic sciences" (Makdisi, 1981, p.9) had to the science of mathematics. As we have seen, according to al-Ghazālī, mathematics was a kind of neutral practice, one that could be useful since it did not venture into the metaphysical realm. Did jurists then make use of mathematical or algebraic treatises such as the *Fakhrī* to help them sort out the questions raised by the variety of transactions that they faced, given the teeming economic activity of the medieval Muslim world? Was this science of the ancients a helpmate to the most important of the religious sciences? We must reiterate that al-Karajī's work seems only to be destined to his fellow mathematicians and algebra enthusiasts as well as to fellow members of the bureaucracy. In this, he does not resemble his illustrious predecessor in the field of algebra, Mu□ammad b. Mūsā al-Khwārizmī (d.ca. 835-844 CE), who clearly states that his *Kitāb al-jabr wa-al-muqābala* (The Book of Restoring and Balancing [i.e. Algebra]) is a guide to calculation with the goal of offering

> what is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, law-suits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned… (al-Khwārizmī, 1989, p.66)

It is interesting that al-Khwārizmī closes the introduction to his work by hoping that his treatise written for the greater good will earn him the redemptive prayers of those to whom he has been of service. A large portion of the book is devoted to working out problems related to the Islamic law of inheritance that do not conform to typical algebraic exercises. Al-Khwārizmī's treatment of algebraic problems dealing with questions of inheritance was thus truly in the service of the Islamic laws of inheritance and not simply a body of abstract mathematical exercises, making it into an "*algebra* of inheritance" (Gandz, 1938, pp. 327-328). We can perhaps assume that many jurists did make use of his treatise to facilitate the calculations required for Islamic inheritance law. Ahmed Djebbar has noted that several prominent jurists spoke out against the use of algebra because they felt that traditional arithmetic methods sufficed for their tasks. He nevertheless also points out that, in actual practice and in reference to the law of inheritance in particular, several algebra manuals existed for jurists. While some jurists refused to make use of them, others did use them or were content to combine traditional methods with algebraic ones (Djebbar, 2005, pp. 46-48).

The question of the impact of the sciences and of mathematics in particular on the daily tasks of the jurists and individual Muslim believers is one that requires further exploration[16]. In the time of al-Ghazālī, an age when the study of the law was becoming increasingly institutionalized and systematized, we know that the libraries of legal colleges

[16] Rebstock's (2007) plenary lecture at the 5th European Summer University on the History and Epistemology in Mathematics Education dealt with this point, among others.

were equipped with scientific treatises, even if the study of the sciences was not part of the official curriculum (Makdisi, 1981, p.78). We also know that, often, jurists and other religious functionaries did not pay attention to the advances in calculation provided by Muslim mathematicians. They preferred to establish the times for the daily prayers or for the beginning of the Ramadan fast according to their own observations and the "folk astronomy" that had been in place since the time of the Prophet. This resistance to adopting the more accurate methods offered by mathematics may have been the result of a desire to follow in the footsteps of the early religious elite or may have been due to their lack of access to, as well as the complexity of, the scientific treatises of the time (King, 1990). However, as the esteemed historian of medieval Arabic astronomy David A. King asks, what of science in the service of religion? (King, 1990). The eras in which al-Khwārizmī and al-Karajī respectively lived were both flooded with the Pythagorean fascination with disembodied numbers and their quasi-spiritual role in the creation and elaboration of the universe, at least as far as the intellectual elite were concerned (Netton, 1982, pp. 9-10). The mathematician should then have had little time or patience to spend his time working out calculations to facilitate the everyday religious rituals of the common believers, but this was not the case. Many Muslim scientists put their art to the service of their religion by painstakingly elaborating prayer-times and the time for the fast as well as determining the correct direction for prayer, namely, towards Mecca (King, 1990). The spirit of the age truly motivated the science of the age, such that we can say that there were "Islamic aspects of Islamic science" (King, 1990, p. 245) and surely an Islamic mathematics to accompany it. This mathematics might be in the service of the practical demands of the ruler or the administration, the exigencies of the religious law or even the daily religious activity of the individual. In the realm of what U. Rebstock has called "practical mathematics," where the work of the professional mathematician was brought to bear on the daily activities of the jurist and the merchant and vice versa (Rebstock 2007), numbers and their usage took on a significant cultural role. Numbers came to mediate forms of social relations in accordance with a concept of the self, vested, as we saw, with the law of God and an omnipresent ethical dimension with the ensuing normativity of what is meant by "good" and "fair" actions.

5. Concluding Remarks

In this paper, we dealt with the problem of the relationship between mathematics and culture. In tune with ethnomathematics, we claimed that mathematical objects are not pre-existing entities but rather conceptual objects generated in the course of human activity. Our goal here was not to account for the manner in which mathematical objects are generated in a given culture but to investigate the social role of mathematics in the creation of new cultural forms of understanding and novel forms of subjectivity. It is our contention, indeed, that mathematics is much more than just a form of knowledge production –an exercise in theorization. If it is true that individuals create mathematics, it is no less true that, in turn, mathematics affects the way individuals are, live and think about themselves and others. Another way of saying this is that, in the act of knowing, subject and object are mutually constituted. There is a dialectical relationship between knowing and being: *Knowing something* is at the same time *being someone*. But the way this mutual constitution occurs is not the mere effect of a two-term relationship (a relationship between subject and object). It is mediated by –and only possible through– social praxis.

Social praxes, we suggested, have a fourfold pyramidal structure, where the vertices interact amongst themselves in a continuous and transformative manner. Forms of social relations and production are entangled with the available cultural knowledge and with ontological cultural conceptions about the way the world *is* and can be investigated. Considered from this fourfold pyramidal structure , both *objectification* (as a process of knowing) and *subjectification* (as a process of being or becoming) are intertwined with webs of social relations and technologies subsumed in cultural modes of acting and reflecting – networks of *ethics* and *truth*.

It was within this context that we examined the role of mathematics in the shaping of subjectivities and understandings. One of the claims that we made was that mathematics creates conditions for certain forms of subjectivity and understanding to arise. Our claim is qualified by a general view of mathematics. According to this view, mathematics is conceptualized as a higher form of reflection of cultural reality –something that is both a form of understanding and sense-making, and a mode of actual and/or potential action. Within this line of thought, in the most general terms, mathematical objects are intellectual or cognitive tools that allow us to reflect upon and act in the world –be it at a concrete level or an abstract, generalized one (from concrete counting to abstract commutative groups, from experiencing motions to differential equations, from our phenomenological surroundings and its embodied ideas of e.g. distance to topological spaces, etc.).

In the two case studies, mathematics appeared as a tool which could be used to reflect, act and situate oneself within a cultural context. We see how mathematics affects all of society and not only those who practice it in a professional way. Thus, the mathematics of the medieval Islamic world, as we saw, was a key element in culturally defining how labour and fair remuneration had to be understood. Mathematics, even for those who were not versed in calculations, affected the manner in which they had to pay taxes, the times for the daily prayers, the time for the beginning of Ramadan, etc. We also saw that in the Late Medieval and Renaissance periods, mathematics was instrumental in creating cultural forms of sensitivity reflected in practical and artistic matters. Mathematics was also central in the constitution of what, with Max Weber (1992), can be called an "instrumental (or practical) rationality" and a concomitant subjectivity that was very different from those of the previous periods. We also saw that this self-centered subjectivity clashed with the ethics of the Church and became, for some, a painful and untenable experience.

The Renaissance indeed marks the moment in the Western tradition when, for the first time, the self extracted itself from its community and put its own interests before anything else. It was at this moment that the idea of the self of modern times emerged – i.e. the notion of the person as bounded, unique, integrated; a self-sufficient dynamic centre of judgement and action (Shweder & LeVine, 1984).

If it is true that the modern concept of subjectivity is poorly equipped to deal with a multi-ethnic world and to interact in a sensitive way with other cultural traditions, and if it is true that we cannot undo history, we can at least learn from it and attempt, as Ubiratan

D'Ambrosio suggests, building a better world in which to live for ourselves and, most of all, for those to come.

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