

DISSIMILAR WAYS OF INSCRIBING SIMILAR TRIANGLES

Ivor Grattan-Guinness†
Middlesex University Business School – United Kingdom

Resumo

This paper contrasts four different ways of handling similar triangles in Euclidean geometry, especially inscription: using 1) elementary geometry, 2) the so-called ‘modern geometry’, 3) a particular theory of ‘geometrical transformations’, and 4) a very general conception of polygons that also studies the letterings of the vertices of polygons. While all of these methods are known, the first is treated more systematically than usual while the latter three are not as well recognised as they deserve to be. Some seemingly new properties of similar triangles are presented.

Keywords: similar triangles, modern geometry, geometric transformations, n-gons, teaching geometry.

[FORMAS DISSIMILARES DE SE INSCREVER TRIÂNGULOS SIMILARES]

Resumo

Este trabalho contrasta quatro maneiras diferentes de lidar com triângulos similares em geometria euclidiana, especialmente inscrição: usando 1) geometria elementar, 2) a amplamente comentada ‘geometria moderna’, 3) uma teoria particular de ‘transformações geométricas’, e 4) uma concepção bastante ampla de polígonos que também estuda os letramentos dos vértices de polígonos. Enquanto todos estes métodos são conhecidos, o primeiro é tratado mais sistematicamente que o usual, enquanto os últimos três não são tão bem conhecidos como merecem sê-lo. Algumas propriedades aparentemente novas de similaridade entre triângulos são apresentadas.

Keywords: Triângulos similares, geometria moderna, transformações geométricas, n-ângonos, ensino de geometria.

1. Introduction

Ever since Euclid presented a formidable body of theorems about polygons and circles, plane geometry has been a fruitful source of further results. In addition, a variety of types of proofs gradually emerged: not only those based upon Euclid's own repertoire but also some that required more complicated geometrical constructions, and those that drew upon other branches of mathematics such as the calculus and various algebras.

In this paper we exhibit a range of seemingly new results that relate a planar triangle $\triangle ABC$ to similar triangles $\triangle PQR$ inscribed within it according to certain criteria. The study was undertaken in the hope of emulating a recent examination of properties of rectangles that are inscribed inside rectangles [Grattan-Guinness 2012]. The proofs given here are geometrical, some of a Euclidean kind (section 2) but others of a more elaborate cast (section 3). Then two types of proof that use algebras are aired; one of them also formulates parts of the underlying logic (sections 4 and 5).

Throughout the mathematics is suitable for teaching purposes; several points for classroom discussion or potential exercises are mentioned. Some relevant books and papers are cited and discussed, but no comprehensive literature review is attempted.

2. Using elementary Euclidean geometry

2.1 *Configurations.* 'Similar triangles' means that each angle in one triangle is equal to one in the other. $\triangle PQR$ is 'totally inscribed' in $\triangle ABC$ if it lies entirely within it, with at least one of its vertices not lying on a side. We can construct such a triangle by drawing its own sides inside and parallel to those of $\triangle ABC$. It can also be moved around within $\triangle ABC$ to some extent; the various triangle inequalities [Nelson 2008] are useful.

We do not discuss this kind further, but focus upon cases where P, Q and R lie *on* the sides AB, BC and CA, and moreover on their *interiors* and not at their end-points A, B and C themselves. We confine the discussion to this "narrow" sense of inscribing $\triangle PQR$; it could be extended to cover the "broad" sense where some of those vertices coincide with A, B or C or lie upon the extensions of the appropriate sides of $\triangle ABC$. These distinctions are customarily made in this kind of geometry.

We associate P, Q and R with the respective angles $\angle A$, $\angle B$ and $\angle C$. There are six different ways of locating these vertices on the sides of $\triangle ABC$, three with P, Q and R oriented clockwise and the other three read anticlockwise. Figure 1 shows examples of the six configurations for $\triangle ABC$ and the orientations. Each case is labelled in the style '(RPQ)', which indicates the angles that are subtended by the vertices on AB, BC and CA *in that order*. $\triangle ABC$ is composed of the "inner" similar $\triangle PQR$ and three non-similar "outer" triangles. If we reflect each configuration about one of the sides of $\triangle ABC$, then we obtain the sextet that pertains to the reflexively congruent triangle to $\triangle ABC$, but no new information is obtained.

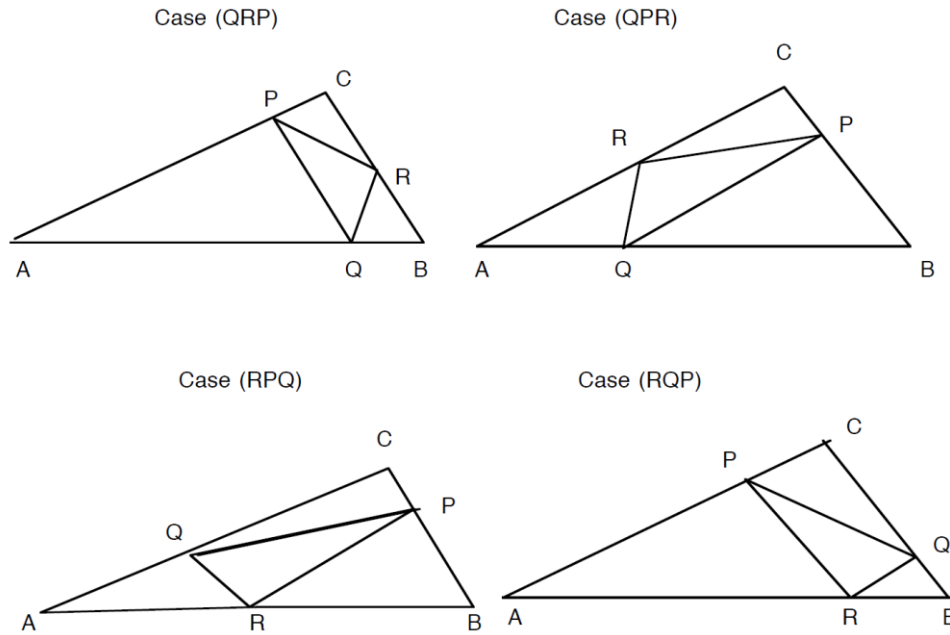


Figure 1.

2.2 *When opposite angles are equal.* From now on we concentrate on the case (RPQ) that stipulates that equal angles in $\triangle ABC$ and $\triangle PQR$ be placed opposite each other:

$$\angle P = \angle A, \angle Q = \angle B \text{ and } \angle R = \angle C.$$

Similarity entails that the corresponding sides of the two triangles are the same ratio (T, say):

$$T = PQ/AB = QR/BC = RP/CA < 1. \quad (1)$$

Each of the other five cases can be treated by the methods about to be described.

One way to determine $\triangle PQR$ is to assume the point A and the directions AB and AC and set through A a line Ln in any direction that lies outside the sector embraced by the two directions and thus cuts both AB and AC when it is moved parallel to itself. Choose any position of Ln to define Q and R as its respective points of intersection with AB and AC, and find P as the point of intersection of the lines inside the sector from Q set at $\angle B$ to QR and from R set at $\angle C$ to QR. B and C are found by placing a line through P and rotating it until we find either the point B on the extended AR while $\angle ABC = \angle B$ or the corresponding point C on the extended AR where $\angle ACB = \angle C$.

This construction works backwards, specifying $\triangle PQR$ and finding $\triangle ABC$; when read in the proper reverse order, we realise that it is not always possible to construct $\triangle PQR$. In the example just described, $\angle I$ must be small enough for BQ to intersect with CA at all, and $\angle AQR = (C - B + I)$ large enough for QR to meet CB.

Assuming that the construction is possible, we specify the chosen direction Dn of (say) PQ by $\angle CPQ$, which we name $\angle I$. Given this angular measure between ΔPQR and ΔABC and the choice of angles to be placed at its three vertices, ΔPQR is *unique* relative to ΔABC and some unit of measure. For each line parallel to PQ and placed above *or* below the one shown in the diagram will lead to the third vertex that indeed lies on CR but is inside *or* outside the ΔABC ; so conditions of continuity suggest only one triangle. Alternatively, ΔPQR is unique for the location of one of the points on a side (for instance, for P on BC) and choice of angles. The scrupulous can specify continuity rigorously by, for instance, the usual Cantorian or Dedekindian procedures.

A variant method is to choose any point R on (say) AB, draw two lines at R with angular separation $\angle C$ and turn the pair of lines around R as centre until the points Q and P are located. Conditions of continuity again guarantee the uniqueness of the construction, although they stand out less clearly than in the previous version.

2.3 *On the circumradii.* We now turn to some properties of these triangles, especially the requirement that PQ subtends $\angle C$ at both C and R.

1) It follows that the circumradii of ΔPQR equals that of ΔPQC ; for the same reason, so do the circumradii of ΔRPB and of ΔQRA . Thus if C is reflected about PQ to move to point C', then PQR C' is a cyclic quadrilateral, to which Ptolemy's theorem applies. If A and B are reflected similarly and separately to create the points A' and B', then we have the same cyclic quadrilateral twice more; and also the cyclic hexagon PA'QB'RC', to which Pascal's collinearity theorem applies.

2) How large is ΔPQR relative to ΔABC ? The size is a function of Dn. Let T be the ratio of corresponding sides on ΔPQR and ΔABC ; setting the circumradii of ΔPQR as r and of ΔABC as U, then $T = r/U < 1$. The areas of the triangles relate as follows:

$$\text{Ar}(\Delta ABC) = \text{Ar}(\Delta PQR) + \text{Ar}(\Delta AQR) + \text{Ar}(\Delta BRP) + \text{Ar}(\Delta CPQ);$$

thus $\text{Ar}(\Delta AQR) + \text{Ar}(\Delta BRP) + \text{Ar}(\Delta CPQ) = (1 - T^2)\text{Ar}(\Delta ABC)$.

The corresponding result for the perimeters is

$$\text{Per}(\Delta AQR) + \text{Per}(\Delta BRP) + \text{Per}(\Delta CPQ) = (1 + T)\text{Per}(\Delta ABC).$$

But further progress is not obvious; in particular, the formula for the area of any triangle ΔLMN with circumradius Y,

$$\text{Ar}(\Delta LMN) = 2Y^2 \sin L \sin M \sin N$$

leads to clumsy trigonometry.

3) A better move is to determine T from (1). Since all four triangles comprising ΔABC have the same circumradius, the sine law states that each side of *any* of the four triangles is in the same ratio to its opposite angle(s); for example,

$$PQ/\sin B = CP/\sin(C + I) = 2TU,$$

while for ΔABC , $AB/\sin C = 2U$. So take, say,

$$1/T = BC/QR = (CP + PB)/QR;$$

$$\text{Then } 1/T = [\sin(\pi - C - I) + \sin(A + I - B)]/\sin A$$

$$= 2 [\sin^{1/2}(\pi - 2I + 2B) \cos^{1/2}(A - \pi)]/\sin A = 2 \cos(I - B);$$

$$\text{hence } T = \frac{1}{2} \sec(I - B).$$

Since $T < 1$, $\sec(I - B) > \frac{1}{2}$, so that $(\angle I - \angle B) < \pi/6$.

The presence of $\angle B$ in this formula does not arise from a special status over $\angle A$ and $\angle C$, but from our decision to specify D_n with $\angle I$ at C ; relative to it $\angle B$ at Q is its alternate. Had we chosen instead $\angle AQR = \angle J$ at Q , for instance, then the determination of T would have used $(J - A)$ rather than $(I - B)$, since

$B + J = A + I$. The relation $A + B + C = \pi$ causes the capacity for multiple specifications of angles.

4) Our configuration complements that of the *orthic triangle* of the acute-angled $\triangle ABC$, that is, the triangle whose vertices are the feet of the three (concurrent) altitudes AL , BM and CN ; for it is known that $\triangle LMN$ is not similar to $\triangle ABC$ but that its three outer triangles are [Altshiller-Court 1952, 97]. Thus if, say, L and M were chosen as P and Q respectively in our configuration, then either R is not N or the construction cannot be completed at all.

5) Among other results, the smallest value of T is given by $\angle I = \angle B$: P , Q and R are the midpoints of their respective sides, the four triangles are congruent, and their corresponding sides are parallel (Figure 2). Again, if $\triangle ABC$ is acute-angled, setting $\angle I = (\pi - 2C)$ makes $\triangle QPC$ isosceles at P , and its neighbour $\triangle RAQ$ along AC is isosceles at R (Figure 3). This property applies also the triangles on AB with vertices Q and P , and on BC with P and Q . In Figure 1 $\angle C$ happens to be obtuse, so this property holds only along AB .

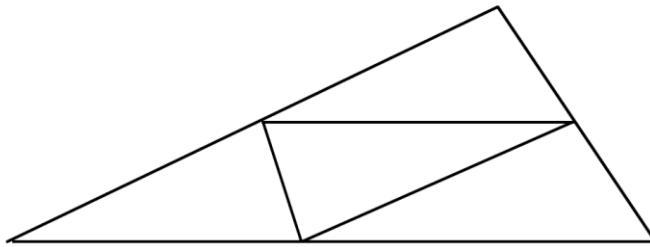


Figure 2: the mid-points triangle.

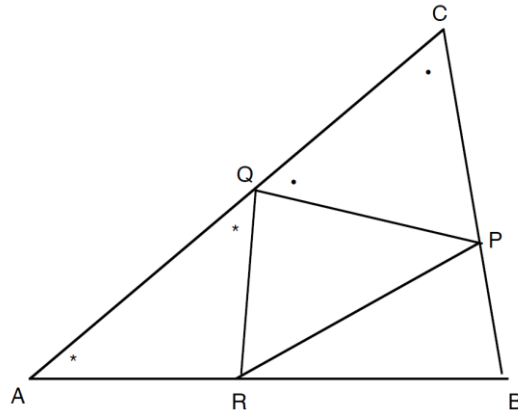


Figure 3: isosceles triangles.

2.4 Extensions.

1) The process of inscribing similar triangles can be *iterated* indefinitely, where the relative linear sizes are determined by the ratios T, T^2, T^3, \dots ; the iteration is launched in Figure 4. The lengths of the sides of the successively inscribed triangles decrease monotonically toward zero, so that ΔPQR moves towards a point. However, the angular positions of the successive triangles do not tend to any final direction. This somewhat unusual limit process obtains in other constructions of this kind, such as the inscription of rectangles inside rectangles.

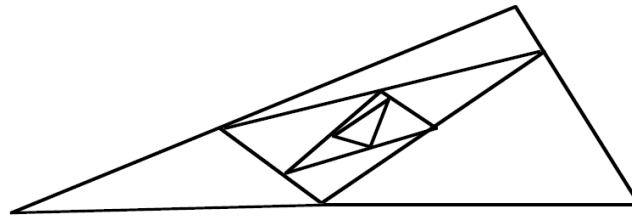


Figure 4: the next two inscribed triangles.

2) ΔABC can also be *circumscribed* in ways corresponding to inscription (again, we specify circumscription narrowly). The diagrams in Figure 1 may be reread as starting with ΔPQR , choosing a line L_n that passes through (say) P in any direction that keeps it outside ΔPQR , and fixing the points B and C on L_n such that $\angle PBR = \angle RQP$ and $\angle PCQ = \angle QRP$. We can start with R or S instead, circumscribe iteratively a finite number of times to create diagrams of sizes T^{-n} , create totally inscribed or circumscribed triangles at will, explore properties of encircles, and work with the reflexively congruent triangle to ΔPQR .

3) The analogous theory in solid geometry concerns the construction of a similar tetrahedron inside a tetrahedron. While there are many theorems about tetrahedra (for example, [Altshiller-Court 1964, ch. 4]), none seems to suggest properties of use here.

3. Using 'modern geometry'

3.1 *Further properties of this case.* From the 1920s and especially in the USA, this name came to be attached to a large body of theorems in Euclidean geometry that are associated with a thicket of special points, straight lines and circular arcs that nestle inside often complicated diagrams: nine-point circle, Simson line, pedal triangle, Apollonios circles, and so on, some named after their supposed creators. Several theorems and constructions are relevant here. We continue with the case of opposite angles being equal.

1) Ignoring similarity, there is the theorem that the circumcircles of $\triangle AQR$, $\triangle BRP$ and $\triangle CPQ$ intersect at a point M that is named after its discoverer [Miquel 1838]. Since two triangles inscribed in $\triangle ABC$ are similar if and only if they have the same Miquel point, M is also that point for $\triangle ABC$, and also for every similar triangle produced by iteration. Conversely, given $\triangle ABC$ and M , there are indefinitely many trios of circles for which M is the Miquel point, and the triangles associated with each trio are similar to each other. A further property of a Miquel point is that the lines from M to P , Q and R make the same angle with the associated sides of $\triangle ABC$; so M is also a Brocard point of $\triangle PQR$.

We saw in section 2.3 that the three outer triangles have the same circumradius r ; so their centres lie on the circle with centre M and radius r (and in any case form a triangle similar to $\triangle ABC$). Figure 5 shows all five equal circles, with the three outer circles drawn with thinner lines. M lies just outside $\triangle ABC$ because $\angle C$ happens to be slightly obtuse.

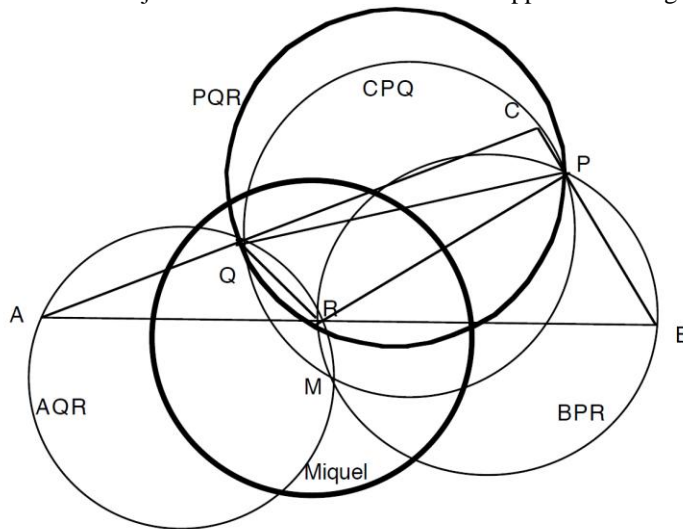


Figure 5: five circles of the same radius.

2) Several other theorems can be taken as embodying methods for inscribing and/or circumscribing (similar) triangles and finding properties of the configurations. For example, Ptolemy's theorem applies also to the cyclic quadrilaterals MQAR, MRBP and MPCQ, which together comprise $\triangle ABC$. Again, further properties arises from drawing and considering AP, BQ and CR; for instance, if AP, BQ and CR are concurrent (at point Z, say), then the two triangles are in perspective with Z as centre and the line at infinity as axis, and iteration produces the attractive process in which the sides of the successive triangles are alternately parallel to those of their parent triangles, and P, Q and R tend down their respective chords towards.

Rich selections of results are provided in [Johnson 1929, chs. 7, 16-17] and [Shively 1934, chs. 2 and 5]. Unfortunately many of them require too elaborate a context for summary or illustration here. Some of them can be extended to allow for apply to the broader senses of inscription and circumscription of triangles that were noted in section 2.1.

3.2 *Unfamiliarity*. Although it overlaps both with elementary Euclidean and with projective geometry, modern geometry has always been rather fugitive in mathematics and mathematics education (see [Romera-Lebret 2012] for an excellent historical summary). For examples from Germany, the school-teacher A. Emmerich produced, for teaching purposes, an excellent survey and bibliography [1891] of 'The Brocard structure and its relationships to the remarkable related points and circles'. Modern geometry also featured quite well in the volume on geometry of the great *Encyklopädie der mathematischen Wissenschaften* (see especially [Burkhan and Meyer 1921] on the 'newer triangle geometry'). However, the director of that work, Felix Klein, gave it little attention in his famous survey of elementary geometry from an advanced standpoint [1925].

A modest tradition developed in the USA for teaching modern geometry. A pioneer author was Roger A. Johnson, with a book [1929] with that title,¹ in which he gave several references. But consider his teacher, J. L. Coolidge. A large survey of 'the geometry of the circle and the sphere' [1916] began with a long chapter on elementary geometry in which this material was prominent, with separate sections on Brocard and Miquel points. However, his later 'history of geometrical methods' [1940] omitted it almost completely. Maybe the reason was that it did not exhibit new methods as such, but he did not use it as source of cases; his bibliography did not even include Johnson.

Modern geometry was encouraged later by books such as [Shively 1934] and [Coxeter and Greitzer 1967], and enriched for solid geometry in [Altshiller-Court 1964]. However, for some reason the bulk of books and textbooks on geometry did or do not give much attention to this branch, often not at all; for example, I cannot recall any mention of it at all in my own school or university education in Britain.²

¹ Note the trendy use of 'modern', evident also in the 'modern algebra' (1930-1931) of B. L. van der Waerden; later the alternative name 'advanced Euclidean geometry' came in, including in the 1960 reprint of Johnson's book. 'Modern analysis' had appeared as a title in the early 1900s, signifying the use of set theory.

² Continued fractions is another of these strangely fugitive topics in mathematics. At present I am working on a similarly fugitive topic, the couple in statics; it was recognised as a basic feature of mechanics only in 1803, by

Coolidge's actions also exemplify that the history of modern geometry has been patchily handled. Max Simon cited many results in his superb bibliographical survey of 'the development of elementary geometry during the 19th century', prepared for the *Deutsche Mathematiker-Vereinigung* [1906, especially parts D and E]; this work headed Johnson's literature review [1929, vi]. In addition, the Scot J. S. Mackay wrote diligently at that time on parts of the history. But these are the exceptions; more normal is a recent distinguished general history of geometry that treats most branches but not this one [Schreiber and Scriba 2010].

4. Yaglom on transformations

As part of a delightful and wide-ranging survey of 'geometric transformations' the Russian mathematician Isaak Moiseevich Yaglom (1921-1988) examined in [1968] 'spiral similarities', which are based upon properties of the equiangular spiral Sp , a curve higher in order than the circle and sphere of elementary and modern geometry. Call its origin O , and choose the direction OM of the tangent at O : then an arbitrary point Z on it has polar coordinates (r, θ) where $r = OZ$ and $\theta = \angle MOZ$. Its equation is given by

$$r = h \exp(\theta \cot k), \text{ where } h \text{ and } k \text{ are constants,} \quad (2)$$

k being the angle between OZ and the tangent to Sp at Z . To fit $\triangle ABC$ on Sp , choose one of its vertices as O and determine h and k by requiring (2) to be satisfied by the coordinates of the other two vertices. In the particular configurations shown in Figure 1, B is a suitable choice for O , and the spiral rotates in either the clockwise or anti-clockwise direction to pass through A and C . In general a spiral is not unique for a configuration.

The exponential character (2) of Sp ensures that it passes through the vertices of not only $\triangle ABC$ but also (by conditions of continuity) the vertices of every inscribed similar $\triangle PQR$ that is produced by iterations as Z slides down to A , thereby effecting the construction [Yaglom 1968, 38 (problem), 119-121]; the iterated inscriptions shown in Figure 4 shows four positions of the triangles as the spiral rotates. This situation can also be read conversely: Sp passes through every similar circumscribed $\triangle ABC$ of $\triangle PQR$. Both processes are prettily represented on the computer by rotations [Thron 2011]; they correspond to case (PRQ) in Figure 1, and Brocard point's feature well.

Further, the anticlockwise spiral dual to Sp deals with the triangle that is reflexively congruent to $\triangle ABC$. The two cases of permuted vertices can be handled, though with different spirals. Yaglom applies it also to some other polygons, including to rectangles in the passages cited here; so his book should be added to the bibliography of my recent paper on inscribing rectangles [Grattan-Guinness 2012]. However, he does not invoke continuity or any other property to guarantee the existence of $\triangle PQR$.

The theory of spiral similarities was inspired to some extent by Klein's conception [1925] of geometries in terms of transformations and the attendant algebraic structures. Part of the general theory of tilings and tessellations, it is summarised in [Coxeter and Greitzer 1967, 95-100].

Louis Poinot in his textbook *Éléments de statique*, which is still one of the best sources to learn the theory; for some reason is omitted from almost all histories of mechanics that cover that period.

5. Bachmann on polygons and their letterings

5.1 *Principles.* The use in geometry of algebras — common, vector, linear, abstract, Grassmann — needs no rehearsal here. But there is an intriguing variant due to the German logician and mathematician Friedrich Bachmann (1909–1982); working with Eckart Schmidt, he elaborated a theory of polygons, which he called ‘n-gons’ [Bachmann and Schmidt 1970].

The starting-point is the theorem that the midpoints of any ‘quadrangle’ QI form the vertices of a parallelogram Pm ; it is easily proved by consideration of similar triangles. Given QI , Pm is unique, but not vice versa; and while Pm is always planar, QI might not be. Similar triangles form an example [fig. 6; also [Bachmann 1971], 290]: given the 3-gon ΔPQR , determine a similar circumscribed ΔABC where P , Q and R are midpoints of its sides.

The theory works with an m -dimensional vector space defined over a field such as the rational numbers, which provide continuity conditions. An n -gon is a closed figure of up to m dimensions in V , specified by 1) a finite set of vertices, each one determined by a (free) vector; 2) the order in which the vertices are taken, with each one joined to its successor by a straight line, called a ‘side vector’; and 3) the choice of a vertex as first in that order, and therefore also the last. The novelty lies in 2) and 3), with a lettering of a figure treated as a kind of mathematical “object”; maybe Bachmann’s training as a logician (under Heinrich Scholz at Münster) led him to consider it.

To each vertex there corresponds just one vector, but the converse is not necessarily the case in that a vertex may belong more than once to an n -gon, which becomes an ordered multiset of vertices. In particular, the centre-of-gravity n -gon of any n -gon N , be it a set or a multiset of vertices, is a multi-unit set containing only the vertex corresponding to the vector defined by the arithmetic mean of all the vertices of N but chosen once for each of them [p. 16].

Particular kinds of n -gon are specifiable. For example [p. 15], the parallelogram is the 4-gon $IJKL$ (in that order of its vertices) of which the corresponding coplanar side vectors satisfy

$$(\mathbf{j} - \mathbf{i}) + (\mathbf{l} - \mathbf{k}) = \mathbf{o} \text{ (the vector zero).}$$

5.2 *Mappings.* The bulk of the theory concerns the mapping of an n -gon onto an n -gon. Of special interest are mappings of an n -gon onto itself but with a reordering of its vertices; in particular, ‘cyclic permutations’ such as of (I, J, K, L) to (K, L, I, J) . Take any n -gon, form the $(n - 1)$ n -gons resulting from every cyclic permutation of its vertices $\{\mathbf{a}_r\}$, and assume that each n -gon exhibits the same linear dependence of its vertices over K ; that is, they satisfy the set of n equations

$$\sum_r c_r \mathbf{a}_{(r+p)(\text{mod } n)} = \mathbf{o}, \text{ where the given } c_r \in K, 0 \leq r \leq n - 1 \text{ and } 1 \leq p \leq n.$$

Then this set of n -gons forms a ‘cyclic class’ [ch. 1]; for example, the four cycles of vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}\}$ of the parallelogram above. A ‘cyclic mapping’ of all n -gons N into n -gons O with vertices $\{\mathbf{b}_r\}$ requires that K contains elements c_r that satisfy

$$\sum_r c_r \mathbf{a}_{(r+p) \pmod n} = \mathbf{b}_r, 0 \leq r \leq n-1,$$

[ch. 2]. The n -gon defined from a given n -gon as containing the same vertices but taken in the reverse order has a theory of ‘anticyclic classes’ [ch. 12]. The vertices of an n -gon may be mapped onto points on another n -gon that are not among its vertices, such as the mid-points of the sides of QI in the starting-point above.

Mappings can be iterated and compounded, and the attendant properties studied in operator algebra. An important example of iteration is any ‘idempotent’ mapping f of a set (not necessarily an n -gon) that satisfies the Boolean law $ff = f$; it is called a ‘projection’. In particular, cyclic mappings of n -gons are cyclic projections because the permutation of a permutation is a permutation [ch. 5]. Idempotence occurs also in the optimal elements 0 and 1 of an algebra (where $00 = 0$ and $11 = 1$). The ‘main theorem’ of the theory is that the mappings of the cyclic classes of n -gons form a finite Boolean algebra, which is a sublattice of the lattice of the sub-spaces of V [ch. 6]. Properties are proved about many kinds of n -gon, and the mappings to which they are subject. Those involving similar triangles are not highlighted, but they are expressible in the theory, indeed in enriched forms that take accounts of letterings. For example, relabelling the six cases in Figure 1, and other such cases, could be undertaken by treating permutations as operators.

The theory made some impact on publication. Bachmann and Schmidt wrote their book in German, which appeared in 1970: the English translation appeared in 1975, and meanwhile Yaglom had edited the Russian translation of 1973. There was some educational stimulus for the theory; both authors published on it in journals on mathematics education, especially [Bachmann 1971]. But since then interest in it seems to have declined; perhaps the frequent dominance of the algebras over the geometry deterred potential students. So it joined modern geometry, in the shadows.

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Ivor Grattan-Guinness†