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ON EUCLID'S FIVE POSTULATES

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Abstract

The five postulates of Euclid's *Elements* are meta-mathematically deduced from philosophical principles in a historically appropriate way and, thus, the Euclidean *a priori* conception of geometry becomes apparent.

Keywords: Mathematics and Philosophy, History of Geometry, Euclid, Postulates.

[SOBRE OS CINCO POSTULADOS DE EUCLIDES]

[Resumo]

Os cinco postulados dos *Elementos* de Euclides são meta-matematicamente deduzidos a partir de princípios filosóficos de uma forma historicamente apropriada e, deste modo, a concepção euclidiana *a priori* de geometria torna-se aparente.

Palavras-chave: Matemática e filosofia, História da geometria, Euclides, Postulados.

Introduction

Euclid's *Elements* served for over 2000 years as advanced text-book in western and eastern education. There is a long history of understanding and interpreting the definitions, postulates and axioms (of size) of the *Elements*, which tapered off in the 20th century.[1] In a recent note John A. Fossa set the first three postulates in relation to the ancient practice of surveying.[2] After a platonic principle which is explained in the simile of the line (Plato, *Republic* 509dff.) a good pedagogical systematization of a science should display all levels of reality from practical applications over scientific reasoning to philosophical justification. In this article we will add the meta-mathematical justification of all five postulates to achieve a more complete picture of them.[3]

A Systematic Approach

In the *Elements*, Euclid separated the conceptual characterization of geometrical objects from the rules which can be applied to construct new and several instantiations of one and the same kind of objects. As Aristotle mentioned explicitly there are many instantiations of one single notion of a mathematical object according to Plato.[4]

In an *a priori* conception of geometry, the transition of notions to the realm of plurality has to be arranged according to the priority of the notions. Thus, the ideas should be instantiated as accurately as possible. Therefore additional conceptual content should be added to the instantiations as little as possible and, in particular, it should not be added arbitrarily, i.e. not only to some instantiations of the same notion. Thought as intelligible matter, the layer of geometrical construction should only sustain the concepts but remain in the background as is the case in Plato's *chora*.[5]

The notions of the geometrical objects are indicated in the definitions of the *Elements*, but have to be explicated carefully in the light of an ancient philosophy of science, as will be explained in the next section. Important for our purpose is the distinction of *conceptually complete* definitions and *incomplete* definitions. A definition of a geometrical object is conceptually complete (in our sense) if no further species of this object can be distinguished. The definition totally determines this kind of geometrical object except for size and position, which are aspects of the instantiations. For example, the straight line and the right angle are completely determined geometrical objects, as is the circle but not the triangle. There are equilateral triangles, which are again completely determined, but then there are also endless kinds of isosceles triangles, not to mention the triangles with three different sides.

Every geometrical object, apart from the point, has geometrical objects as parts which become important in its definition. E.g., a circle is defined by its center and distance and so has a point and a straight line as parts. For an instantiated point it is a property to be a certain part of a greater whole, which, in our case, is to be the center of a circle. It is a

[1] Although there is a small academic discussion (HEATH, Mueller (1981), Netz (1999), Vitrac (1990), Acerbi (2007), et al.)

[2] Cf. Fossa (2013).

[3] A more formal and detailed treatment of the following argumentation can be found in Schneider (forthcoming). That mathematical axioms are in need of a philosophical justification is a thought ancient philosophers and scientists are familiar with, e.g. Plato *Republic 510c-511d*.

[4] *Metaphysics 987b14-18*.

[5] Cf. *Timaios 49a, 52ab*; Proclus (1970), p.64(78); Aristotle *Metaphysics 1035b31-36a12*.

conceptual content which possibly allows the instantiated points which are centers of certain circles to be distinguished from instantiated points which are not centers of circles.

The core argument of our justification of the postulates now runs as follows in the case of points: If there are points which are endpoints of straight lines, then there has to be a straight line between any two (different) points. *Otherwise one can distinguish between the pairs of points which are endpoints of straight lines and those which are not*. In this case an additional conceptual content would be arbitrarily adjointed to some points. — To make it work, the argument has to be complemented by something like the existence of some instantiations of the particular objects. In the following section we will tacitly assume the existence of a sufficient number of instantiations.

Looking at the postulates from the perspective of the definitions, they appear to be simply inversions of the definitions. Whenever the objects exist which can figure as particular parts of a more complex object, then the whole object exists, whereas a definition states that every particular complex object has these particular objects as parts.

In fact, this thought is not correct in the case of points, because points have no geometrical parts. However, to let points be part of geometry, Euclid related them in definition three to lines: they are their endpoints. The already stated argument for the first postulate involves straight lines as compensation for the non-available parts of points. In the case of right angles there is something different. Right angles are only defined pairwise. Thus, it is *a priori* unknown whether two instantiated right angles of different pairs would add to two right angles making a straight line – or not.

A different problem occurs with the triangle. It is the only nonconceptually complete defined object, which should be implemented in the postulates. The reason for choosing it despite of its incomplete definition is that it is the first straight-lined figure, which in fact is fundamental for all further straight-lined figures. The incompletion, though, makes it necessary to recur to an essential property of all triangles which is only implicitly contained in the Euclidean definitions.

To sum it up, we have the following arrangement of postulates and geometrical objects:

postulate	**considers definition of**	**assumes instantiations of**
1	point	(finite) straight line
2	straight line	(finite) straight line
3	circle	circle
4	right angle	(at least two differently given) right angles
5	triangles	arbitrary triangles

The Essential Properties

Contrary to modern Hilbert-style axiomatizations, the use of philosopical terms is allowed in Euclid's axioms. Notions such as whole-parts, middle-ends, rest-motion, and sameness can be employed to define the geometrical objects and their construction.[6] We will supplement the philosophical input as far as required for the clarification of the definitions.

> *"1. A **point** is that which has no part.*
> *2. A **line** is breadthless length.*
> *3. The extremities of a line are points".* (HEATH, 1956, p. 153)

To define the simplest object as being the first one in a directed sequence of propositions is only possible by negation of what comes later. Thus, points do not share with all other geometrical objects what these have in common: that is to have parts and being a whole. In contrast, points are exactly the objects which have no parts.

The second definition has to be read in the context of the first definition: Lines have parts in opposition to points. But in contrast to surfaces, they have parts only in one way (in length), not in two ways (length and breadth). Contrary to our modern set-theoretic understanding, in this case parts have to be thought of as being of the same type as the whole. Parts of a line are lines again.

In the third definition the objects of the first two definitions become related. A finite line has two endpoints, if it has ends at all.[7] Since we think that Euclid does not employ an actual infinite straight line in the *Elements*, we will skip the adjective "finite" in the following argumentation. Furthermore, these definitions suffice to define the concept of 'lying on a line': A point lies on a line, if and only if the point is the endpoint of a part of the line (or one of the endpoints).

> *"4. A **straight line** is a line which lies evenly with the points on itself".* (HEATH, 1956, p. 153)

This definition is notoriously difficult to understand. We will only explicate which properties will be required in the next section and can possibly be found in this definition, if it has any serious meaning at all.[8] First of all, there is only *one* straight line between two points, which is possibly related to the meaning of "lies evenly". Second, as a line, it has parts and these parts are straight lines again. We draw a distinction between parts of a straight line which have one endpoint in common with the whole straight line (end part) and the parts which are completely within the whole straight line (inside part).

> *"10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands".* (HEATH, 1956, p. 153)

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[6] See Schneider (2012), section 1.3.7, and Schneider & Roth (forthcoming).

[7] We infer the twoness of the endpoints from the analogy with the notions arranged in Plato's *Parmenides 137d*, this is the third notion in the row of the first hypothesis and this is the notion of beginning-middle-end; see Schneider (2012), p.28.

[8] For an attempt to improve our understanding of the fourth definition see Schneider and Roth (forthcoming).

The right angle is defined pairwise. It can be constructed with the help of the first three postulates (*Elements* I, Th.11). However, this does not settle the need for a postulate, because the construction goes along without regarding the quantity of the right angle independent of that particular pairwise construction. Analogous is the definition of the circle: Given a point and a distance, a circle around this point exists *with a certain quantity as radius*. Given a straight line set up …, then there exists a right angle 'supervening' on the two straight lines *with a certain quantity* (of angle), which does not have to be given because there is only one quantity for right angles. — Thus, on the one hand a different kind of angle has a different quantity. On the other hand quantity is nothing which can be used to define an Euclidean geometrical object.[9] The constructive condition does not guarantee the equality of all quantities of right angles of different constructions.

> *"15. A* ***circle*** *is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;*
> *16. And the point is called the* ***centre*** *of the circle".* (HEATH, 1956, p. 153/154)

The center and the distance given by at least one straight line are parts of the circle. There is a radius, because by definition 2 the circumference has parts, which have endpoints which lie on the line, and by postulate 1 there is a straight line between the center and such a point on the circumference.

Triangles are defined as figures with exactly three straight lines as perimeter, and then they are subdivided in further species.

> *"19.* ***Rectilineal figures*** *are those which are contained by straight lines,* ***trilateral*** *figures [triangles] being those contained by three,* ***quadrilaterals*** *those contained by four, and* ***multilaterals*** *those contained by more than four straight lines.*
>
> *20. Of trilateral figures, an* ***equilateral triangle*** *is that which has three sides equal, an* ***isosceles triangle*** *that which has two of its sides alone equal, and a* ***scalene triangle*** *that which has its three sides unequal".* (HEATH, 1956, p. 154)

The definition of the triangle is conceptually incomplete. There are infinitely many conceptually different triangles, hence there cannot be a finite set of definitions defining them. But given any specification of a triangle by the length of its sides, it can easily be constructed similar to the way proceeded in the first theorem. However, there is not a

[9] At first glance, Euclid seems to employ claims about quantity in the definitions. Nevertheless, I think that the equality claims can be reduced to identity, because equality minus position is identity. And the 'greater' and 'less' of the obtuse and acute angle can be reformulated with the concept of whole-and-parts. If this is right, why did not Euclid himself do it this way? Because the *Elements* are a pedagogical, thoughtful beginners book, not a *Principia Mathematica*, which almost nobody read.

triangle for every triple of straight lines. Thus, the explicit Euclidean definition alone cannot be transformed into a postulate. We need to reveal a not explicitly mentioned characterization of the triangle.

If one looks at the overall structure of the Euclidean definitions, one can recognize an unfolding of structure. Starting with the simplest object, the point, the definitions get mostly more complex.[10] In fact, from the perspective of simple modern topology, one is aware of concepts like boundary, side, inside-outside etc., and can recognize an awareness in the definitions of such relations. In this sense, stressing the existence of a center in the definition of circle, is important, because it is only by the center that one can determine the inner area as inside. Otherwise there would be a circumference and two seperated but not determined surfaces only. We call these findings part of an Euclidean proto-topology.

But then, what is the proto-topological characterization of the triangle? We have to look at the definitions preceding definition 19!

> *"17. A* ***diameter*** *of a circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.*
>
> *18. A* ***semicircle*** *is a figure contained by a diameter and the part of the circumference cut off by it. And the center of the semicircle is the same as that of the circle".* (HEATH, 1956, p. 154)

The need for definition 18 is not obvious, but it is useful to build a concept of 'side of a straight line'. Together with the first three postulates one can define what it is to lie on one side of a straight line: *An arbitrary object lies completely on one side of a straight line, if and only if there is an extension of that line, two semicircles having this extended line as diameter, and the object lies completely in one of these two semicircles.*

Notice that the arbitrary extension of the straight line seems to be necessary and is in accordance to the fact that all finite straight lines which are part of the same infinite straight line share the sides. Though, more important is the strange *addendum* to definition 17: "… and such a straight line also bisects the circle". It lacks a proof and its mathematical benefit is vague. We interpret it in the light of the modern debate about indescribability of mathematical objects as saying something about the possibility of making distinctions. And in the light of definition 18 it is something about the sides of the line: *The two sides of the diameter are not (yet) distinguishable.* As a meta-mathematical statement it is not mathematical derivable, but can only be inferred meta-mathematically from the evenness of the straight line and the circle.

Thus, up to definition 18 there is not sufficient structure to distinguish the two sides of a diameter or, more simple, of a straight line. **The triangle in definition 19 is the first geometrical object in the sequence of definitions which makes the sides of its straight lines distinguishable:** *One side of each of the three straight lines of the triangle*

[10] For some more ordering principles of the definitions see Schneider (2012), p.29, Schneider & Roth (forthcoming).

can be characterized as the side on which the other two lines intersect. Therefore, we take as proto-topological characterization of the triangle the property to make the sides of three straight lines distinguishable. Naturally, this only works if the point of view is changed from straight lines determined in their length to indefinitely extended straight lines. *Three different indefinitely extended straight lines form a triangle, if and only if their sides are distinguishable – through the constellation of the other two lines.* We take this as the essential and proto-topological description of the triangle.

Consequently, the main argument will be that three straight lines which make their sides determinable, though they are arbitrarily extended, will have to form a triangle. The condition 'under arbitrary extensions' is also added to strengthen the property in accordance to the way 'lying on one side of a straight line' was explained. Without this condition a simple constellation such as in Figure 1 would determine the sides of all three lines. For each straight line, 'the side on which the other two lines lie' can be used as definite designation of one side.

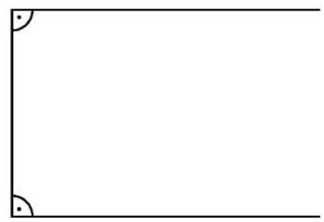

Figure 1

Under extension we get Figure 2 where the sides of only two lines are determinable at first glance.

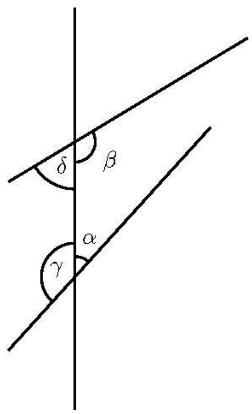

Figure 2

However, if there are different sums of interior angles ($\alpha+\beta\neq\gamma+\delta$), then the sides of the third line are also determined. We have to postulate the existence of a triangle, if the three straight lines were indefinitely extended. If the sums are the same, then they are equal to two right angles and no side is determined. Other possible settings are: 1) Three parallels.

Then the sides of the line 'in the middle' are not determined, because the other two lines are not conceptually distinguishable.

2) Three straight lines, (at least) two of them crossing. If one can achieve a determination of all sides by the first four postulates, then the two crossing lines have to meet the third in two points and form a triangle – if we apply our argument. However, in the postulates only the constellation of Figure 2 is considered, which finally suffices to decide case 2).

The Justification of the Postulates

In the second section the argumentative structure based on a certain homogeneity principle was elucidated, and the definitional content of the geometrical objects was explicated in the last section. In the following the postulates will be derived.

Postulate 1: A straight line has two endpoints. For points to be conceptual 'equal' or 'homogeneous' as characterized in the second section there has to be a straight line between any pair of given points.

Postulate 2: A straight line has (end / inside) parts which are again straight lines. The straight lines which are a (end / inside) part of a larger line should not be separable from possible straight lines which are not (end / inside) parts of a larger straight line. Thus, any straight line has to be (end / inside) part of a larger straight line, and consequently any straight line is indefinitly extendible in both directions.

Postulate 3: A circle has a centre and a straight line between the centre and the circumference. Thus, every endpoint of a given straight line has to be the centre of a circle with the other endpoint lying on the circumference.

Postulate 4: A right angle can be split up into two straight lines in such a way that, if one is continued, they will build equal angles. Thus, all angles with this property have to have the same quantity, because otherwise one could distinguish some pairs of right angles according to their quantity. Therefore, every angle with that constructive property which we call right angle, has to be the same angle, and so the right angle has to be the same angle everywhere.

Postulate 5: A triangle is a figure with exactly three straight lines as perimeter and is the geometrical object with the property that any two of the three straight lines make the sides of the third determinable. Two lines of a triangle (indefinitly extended) meet on one side of the third, which is therefore determined, because they all form a triangle.

If two straight lines cut a third in different points, the sides of the two lines are determined by the point of intersection of the other two lines. If they, in addition, determine the sides of the third straight line by having different sums of interior angles, then the two straight lines have to meet on one side and so form a triangle due to our basic argumentation). According to theorem 17 of the *Elements* proved without the fifth postulate, this is the side with the smaller sum of the interior angles.

Thereby, it seems that we have deduced all five postulates:

> *"Let the following be postulated:*

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles". (HEATH, 1956, p. 154/155)

Conclusion

If one takes the Euclidean definitions as given and presupposes some Platonic meta-mathematical thoughts, one can justify and – in a wide sense – derive all five postulates.[11] Since all assumptions are historically adequate, it is possible that Euclid himself justified and perhaps 'found' the postulates in a similar manner.

Having shown that the axiomatization of the *Elements* has possibly more systematic structure than previously thought, questions about the recognition of consistency and independence of the postulates naturally arise. As is widely known, modern formalizations of elementary Euclidean geometry are consistent and entail no contradiction, and the parallel postulate is independent of the others. But what could Euclid have known about that in a historically appropriate reformulation? Is there something about our method of justification that endows a fresh outlook at the independence and consistency of the postulates, especially the fifth? The answer can be "yes, partly", if there really is a kind of implicit proto-topology in the Euclidean definitions, because this proto-topology shows a continued growth of structure in the course of the definitions. If one further keeps in mind that due to the figure-theoretic approach of ancient geometry, structure is transformed to the postulates only by figures, then the additional structure of the fifth postulate cannot be reached by the other four postulates. However, this is a topic of further research.[12]

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[11] Symmetry and homogeneity occur throughout the history of geometry and the reception of the *Elements*. However, closely related to our approach is mainly Lachterman (1989), p.119, but also see Lorenzen (1984).
[12] I would like to thank Gerhard Kowol, Renatus Ziegler, John A. Fossa and Vincenzo de Risi for helpful comments on another version of this paper. I am also grateful to the Alexander von Humboldt Foundation for financial support and to Hannes Leitgeb and the MCMP for the warm reception. Special thanks to K. P.

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