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CARATHÉODORY AND THE AXIOMATIZATION AND ALGEBRAIZATION OF MEASURE THEORY IN THE FIRST HALF OF THE XX CENTURY.

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Abstract

We give an account Carathéodory's axiomatization of measure theory and his lesser known work on the subject developed in the 1930's. We show how this led him to an algebraization of measure theory.

Keywords: Axiomatization, Measure theory, History of Mathematical Analysis.

[CARATHÉODORY E A AXIOMATIZAÇÃO E ALGEBRIZAÇÃO DA TEORIA DA MEDIDA NA PRIMEIRA METADE DO SÉCULO XX.]

Resumo

Damos um requento da axiomatização da teoria da medida de Carathéodory e o seu trabalho na matéria, menos conhecido, desenvolvido nos 1930s. Mostramos como isso levou-o a uma algebrização da teoria da medida.

Palavras-chave: Axiomatização, Teoria da medida, História do analise matemático.

Introduction. The work of Jordan, Borel and Lebesgue.

Measure theory is a branch of mathematics with a very interesting history whose origins, as an independent theory, can be traced to the second half of the XIX Century. This clearly does not mean that the notion of measure was nonexistent before this; it simply means that from this point onwards a new theory was born.

The concept of measure as such arose primarily, but not only, from research within integration theory. During the 1880's and 1890's the theory of integration had to deal with certain properties of infinite sets and in particular the discovery of sets that were nowhere dense but had positive outer content showed the relevance that these properties were to have with integration theory, and this in turn swiftly led to the rise of several measure theories.

One of the first theories of this kind was developed by Otto Stolz in the years 1881-1884 when dealing with a problem of arc length, (Stolz, 1884), and soon afterwards, in an independent manner, Georg Cantor introduced his own notion of content in (Cantor, 1884). It is also worthwhile noticing important contributions to the subject by Harnack and Peano in (Harnack, 1880) and (Peano, 1883) respectively. Nevertheless, the first of these theories that focused in depth on developing the link between integration and the "size" of a set was that of Camille Jordan.

In 1892 Camille Jordan published a paper, *Remarques sur les intégrales définies*, (Jordan, 1892) in which he claims that the role that functions have in definite integrals is clear and has been well understood but that the influence that the sets on which the functions are defined has on the definite integral needs to be researched in much greater depth.

Jordan shows that given any set E there correspond to it two fixed numbers E' and E'' which he calls E's inner and outer content.[1] If these numbers coincide the set E is measurable with content equal to $E'=E''$. An important fact that is worth noting is that the content of a set will be a length, an area, a volume, etc. depending on the dimension of the set; that is, the concept of content which Jordan defines is an a priori generalization of the preexisting concepts of length, area, volume, etc.

It is also important to note that the process carried out by Jordan is a constructive one. Jordan considers a set E in the plane and a decomposition of E into squares with sides of length r. He then considers the squares whose points are all interior to E, these squares form a domain S. The squares whose points are interior to E or have points on the boundary of E form a domain $S+S'$. As both S and $S+S'$ are made up of squares their areas are well determined and shall be called S and $S+S'$ respectively. Now, if r tends to zero, Jordan shows that S and $S+S'$ tend to well defined limits a and A respectively. Jordan proves that this procedure is independent of the decomposition chosen for E and hence is able to give the following definitions:

Definition. Let E be a set in the plane. $a=e_i$ will be called its interior content, $A=e_e$ its outer content and we shall say that E is measurable if $e_i=e_e$.

An important consequence of this definition of measurability is that if $E = \bigcup_{i=1}^{n} E_i$, where E_j are disjoint sets, then $\sum_{j=1}^{n} e_i\,(E_j) \leq e_i(E) \leq e_e(E) \leq \sum_{j=1}^{n} e_e(E_j)$ as this implies that if the sets E_j are measurable then so is E. It is important to note, specially in the light of what we shall show next, that this additive property is a finite property.

This finite additivity is of great importance in the theory of multiple integrals. If we let$E = \bigcup_{j=1}^{n} E_j$, Jordan then defines the upper and lower sums of a function f on E as$U =$

[1]The word used by Jordan is *étendue*.

$\sum_{j=1}^{n} M(j)e\,(E_j)$ and $L = \sum_{j=1}^{n} m(j)e\,(E_j)$and shows that if the size of the sets E_j tends to 0, then U and L tend towards fixed limits. In this way $\int f(x,y)dE$ exists if and only if these limits coincide, and in such a case is equal to this common value.

This is how Jordan set Riemann's integration theory within the scope of his new measure theory. It is worth noting however that Jordan's measure theory contrasts greatly with the measure theory that Borel was soon to set forward.

The first point of contrast arises from the fact that Borel's interest in measure theory was very different to Jordan's as his primary motivation came from within the theory of functions of complex variable, and in particular the analytic continuation of functions given as series $\sum \frac{a_n}{(z-b_n)^{m_n}}$where m_n is an integer sequence bounded from above and the series $\sum |a_n|$converges.

Borel presents his theory of measure for the first time in 1898 in *Lecons sur la théorie des fonctions*, (Borel, 1898). To define his own concept of measure he considers only sets in the unit interval as follows:

If a set consists of all the points contained in a countable union of disjoint intervals having total length s, we will say that the set has measure s. That is, if $\{I_n\}$ is a countable family of disjoint intervals contained in [0,1] then the measure of the set $\bigcup_n I_n$will be$m(\bigcup_n I_n) = \sum_n m(I_n)$ where $m(I_n)$is defined as the usual length of the interval. If two disjoint sets have measures s and s' respectively, then their union has measure $s+s'$; if one has a countable family of disjoint sets of measures s_1, s_2,… s_n,… , then their union has measure $s_1 + s_2 + \ldots + s_n +$ ... In other words, if $\{E_n\}$ is a disjoint countable family of sets contained in the unit interval, then the measure of $\bigcup_n E_n$ will be $\sum_n m(E_n)$. Finally, if $E_1 \subseteq E_2$ are two sets whose measures are given, then the measure of $E_2 - E_1$will be given by $m(E_2) - m(E_1)$. Borel then goes on to give the following definition:

Definition. The sets whose measure can be defined given these definitions shall be call measurable.

Borel is careful to note that this does not mean that another definition of the measure of sets cannot be given.

Three important consequences can be drawn from Borel's definition of measurability: the measure of a set is never a negative quantity, a set can have measure zero (even if it has the cardinality of the continuum) and a countable set always has measure zero.

The properties with which Borel defines his concept of measure are also the ones needed to define the operations that enable the construction of new measurable sets; the only (implicit) restriction laid down by Borel is that these operations can only be applied a countable number of times. This in turn means that the cardinality of the set of measurable sets is equal to the cardinality of the continuum.

Now, Borel argues in favour of the measure he has defined (versus a measure defined on a bigger class of sets) by saying that it is crucial that a measure has the fundamental properties he has defined. In other words, it is essential to note that a measure cannot be useful if it doesn't possess certain properties, that in this case have been postulated a priori and it is from these properties that the class of measurable sets is derived. This method contrasts with Jordan's as his method was constructive and not descriptive as Borel's. Borel, (Borel, 1898, p. 48), says that “this way of proceeding is analogous to the methods introduced by Jules Drach in Algebra and the theory of differential equations […] In all cases we proceed with the same fundamental idea: define the new elements that are introduced with the help of their essential properties, i.e., those which are strictly necessary for the reasoning to follow.”

Regardless however, of these different approaches to measure, both Jordan's and Borel's ideas were key in the development of Lebesgue's own measure theory at the turn of the XX Century.

In 1902 Lebesgue published his doctoral dissertation *Intégrale, Longueur, Aire*, (Lebesgue 1902). The goal of the thesis is to give precise definitions of the concepts of definite integral, arc length and surface area. Thus, Lebesgue's interest in developing a measure theory coincides with that of Jordan as it originated within integration theory.

Lebesgue introduces a new measure (different to both Jordan's and Borel's) in the same manner as Borel, i.e. by setting out first the properties that need to be met. He thus formulates the measure problem:

> ***Problem 1 (Measure problem).*** *“We intend to attach to each bounded set a measure satisfying the following properties:*
> *1. There exist sets of non-zero measure.*
> *2. Two equal sets have the same measure.*
> *3. The measure of the sum of a finite or infinitely countable number of sets, pairwise without common points, is the sum of the measures of these sets.*
> *We shall solve this measure problem only for the sets that we call measurable.”*(Lebesgue, 1902, p. 236)

These three conditions can be thought of as axioms (or as fundamental properties using Borel's words) that a measure has to meet. We can see that Lebesgue's measure problem thus coincides with Borel's outset but Lebesgue then goes own to construct a measure and prove that it satisfies these properties much in the same way that Jordan constructed his content.

To actually solve the measure problem Lebesgue first solves the problem of uniqueness. It is clear that any multiple of a solution will also solve the measure problem and Lebesgue sets the measure of an arbitrary bounded interval equal to 1; he then observes that if the unit interval is chosen then the measure of each bounded interval will coincide with its usual length. This clearly solves the uniqueness problem Lebesgue mentions but it is worthwhile to note that this does not solve the uniqueness issue in a wider sense.

Lebesgue shows that as consequences of the conditions set out for the problem one will have that any set that contains a single point will have measure zero (as bounded sets with an infinite number of points must have finite measure) and that measure is always non-negative.

The process carried out by Lebesgue to actually construct a measure that will solve the problem set out is as follows: Let E be a bounded set; the points of E can be enclosed in a finite or countable collection of intervals in infinitely different manners. Let E_1 be the set of points of one of these collections of intervals. The measure $m(E)$ of E will be at most equal to the measure of E_1. The infimum over all possible collections is thus an upper bound for $m(E)$ and defines its outer measure $m_e(E)$. In other words, $m_e(E) = \inf\{m(E_1): E_1 = \bigcup_n I_n, I_n \supseteq E \text{ and} I_n \cap I_m = \emptyset \text{ if } n \neq m\}$ where I_n denotes an interval.

Now, if all the points of E belong to a certain segment AB, then the measure of AB-E is at most $m_e(AB\text{-}E)$ and hence the measure of E is at least $m(AB)$-$m_e(AB\text{-}E)$. This number will be called the inner measure of $E$and will be denoted by $m_i(E)$. It is easy to see that $m_e(E) \geq m_i(E)$ is always the case. When equality holds Lebesgue will say that:

Definition.E is a measurable set if $m_e(E) = m_i(E)$ and its measure, $m(E)$ will be equal to this common value.

Given this definition it is then shown that if E_1, E_2,... is a countable collection of measurable sets, then $\bigcup_{n=1}^{\infty} E_n$and $\bigcap_{n=1}^{\infty} E_n$ are also measurable. It is again important to note that it is via these two operations applied a countable number of times that new measurable sets can be obtained. The difference with Borel's procedure is that Lebesgue does not limit himself to intervals to begin with and hence the class of Lebesgue measurable sets is larger that that of Borel measurable ones. In fact, it is easy to see that sets measurable in Borel's sense (sets that will be called B-measurable by Lebesgue) are measurable in the sense of Lebesgue but not vice versa.

> "*[B-measurable sets] are defined by a countable number of conditions, their power set has the power of the continuum* […] *The set E formed by the points with abscissas* $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots$ *where the* a_i *are equal to 0 or 2, being perfect, is B-measurable. Its complement is formed by an interval* $\left(\frac{1}{3}, \frac{2}{3}\right)$ *of length* $\frac{1}{3}$, *by two intervals* […] *of length* $\frac{1}{3^2}$, *by four intervals of length* […] *etc., and hence has measure*[…] *1 and thus E has measure zero. E has the power of the continuum, hence with the points of E an infinite amount of sets, each having exterior measure zero and thus being measurable, can be formed. The power of the set of these sets is that of the set of sets of points; therefore there exist measurable sets that are not B-measurable, and the power of the set of measurable sets is that of the set of sets of points.*"(Lebesgue, 1902, p. 240-241)

On the other hand the difference between the measures of Jordan and Lebesgue can be found when dealing with interior points. If a set has interior points then the measures coincide, however if a set has empty interior then it is not Jordan measurable. In this manner, the set of Jordan measurable sets is also a proper subset of the set of Lebesgue measurable sets, nevertheless in this case one has that "the set of J-measurable sets has the same power as the set of sets of points."(Lebesgue, 1902, p. 242)

These results about the cardinalities of B, J or L-measurable sets easily conduct the reader to wonder if the set of Lebesgue measurable sets coincides with the set of all bounded sets, that is, if any bounded set of reals is Lebesgue measurable. It is important to note however that Lebesgue does not attack this problem at all, he limits his comments on this issue to the following: “It has not be shown that the measure problem is impossible for sets (if they exist) whose interior and exterior measures do not coincide. But in what follows we will only find measurable sets.”(Lebesgue, 1902, p. 239)

The existence of a non-measurable set.

In 1905 Guiseppe Vitali published a paper, *Sul problema della misura dei gruppi di punti di una retta*, (Vitali, 1905), in which he proves that there exists a set of reals which is not Lebesgue measurable. In fact, what Vitali shows is that if the Axiom of Choice (AC)[2] is used then the measure problem has no solution, or in other words, no real, normalized, non negative measure defined on all bounded sets that is translation invariant and countably additive exists: “the problem of measure of groups of points on a line is impossible […] our result implies that the possibility of the measure problem of groups of points on a line and that of well-ordering the continuum cannot coexist.” (Vitali, 1905)

Given the fact that Vitali's proof uses AC, it was not welcomed by Lebesgue who, together with Borel and Baire rejected the use of AC in mathematics.[3]It is important to note that for many other mathematicians, for whom AC was valid, Vitali's proof represented an actual result that was mathematically correct and in fact, it was on this same track that Hausdorff showed in 1914 that Lebesgue's measure problem does not have a solution in n-dimensional space (with $n \geq 3$).[4] To show this, Hausdorff presents the *broad* measure problem: Is it possible to assign to each bounded n-dimensional set E a number $m(E)$ that satisfies the following conditions?

1. $m(E) \geq 0$.
2. $m(E_0) = 1$ for some set E_0.
3. $m(E_1 \cup E_2) = m(E_1) + m(E_2)$ if E_1 and E_2 are disjoint.
4. $m(E_1) = m(E_2)$ if E_1 and E_2 are congruent.

What Hausdorff does (with the aid of AC) is to show the existence of a decomposition of the unit sphere's surface (in three (or more) dimensional space) into four sets A,B,C and Q such that Q is countable, A, B and C are congruent and A is also congruent to $B \cup C$. Hence, if a measure satisfying conditions 1-4 were to exist, it would follow that $m(A) = m(B) = m(C)$ and $m(A) = m(B \cup C)$ which is obviously absurd.

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[2]The Axiom of Choice had been explicitly stated and used by Zermelo for the first time in his 1904 proof that every set can be well-ordered.

[3]For a history of the axiom of choice and its reception by the mathematical community at the beginning of the XX Century see (Hadamard, 1905) and (Moore, 1982).

[4]See (Hausdorff, 1914) Hausdorff explicitly mentions that the problem remains open for 1 and 2 dimensional space and in 1923 Banach, in (Banach, 1923) proves that the problem in these spaces does in fact have a solution. For an account of the development of the measure problem from 1902 to 1930 and the work done by Lebesgue, Vitali, Hausdorff, Banach, Tarski, Kuratowski and Ulam amongst others, see (Martínez-Adame, 2013).

Hausdorff's work which questions the existence of more general measures, together with Carathéodory's work during 1914-1918 allowed for great development within measure theory. We believe it is important to stress that Lebesgue measure was a generalization of the concept of length and not a general abstract concept, Carathéodory, on the other hand, had a very different point of view.

Carathéodory's early work in measure theory.

Constantin Carathéodory upon his arrival in Göttingen in 1913 began the study of real functions. His starting point was the theory that had emerged as a result of Lebesgue's integral, and in 1914 he presented his formal measure theory in the article *Über das lineare Mass von Punktmengen - eine Verallgemeinerung des Längenbegriffes*, (Carathéodory, 1914). He begins this article by stating that:

> "*It seemed* [...] *appropriate to me to begin my presentation with a purely formal theory of measurability. A fundamental definition on measurability is announced, that in one sense is more general than the usual one because it applies to sets of points with infinite outer measures* [...] *this definition is much more convenient than the older one.*" (Carathédory, 1914, p. 404-405)

Carathéodory defines an m-dimensional measure for subsets of q-dimensional euclidean space such that for $m = 1$ one obtains linear measure and for $m = q$ one obtains Lebesgue measure. To do this Carathéodory defines an exterior measure via five basic properties that, in fact, play the roles of axioms in his theory. They are the following:

1. To each set A in $\mathbf{R}^q$ a unique number, $m^*(A)$, which can be 0, positive or ∞, is assigned and is called the exterior measure of A.
2. If B is a subset of $A \subseteq \mathbf{R}^q$ then $m^*(B) \leq m^*(A)$.
3. If A is the union of a countable sequence of sets $A_1, A_2, \ldots$ in $\mathbf{R}^q$, then $m^*(A) \leq m^*(A_1) + m^*(A_2) + \ldots$ This inequality holds only if the sum converges.
4. If A_1 and A_2 are two sets in $\mathbf{R}^q$ and the distance between them is positive, then $m^*(A_1 + A_2) = m^*(A_1) + m^*(A_2)$.[5]
5. The exterior measure $m^*(A)$ of any set A in $\mathbf{R}^q$ is the infimum of $m(B)$ taken over all measurable sets B that contain A.

After the first three of these axioms Carathéodory presents the definition of a measurable set:

Definition. A set A is measurable if for any subset W, one has $m^*(W) = m^*(A \cap W) + m^*(W - A \cap W)$. The measure $m(A)$ of A is then defined by the equation $m(A) = m^*(A)$.

It is important to note that after introducing this definition Carathédory proves various theorems and only after doing so introduces axioms 4 and 5.

[5] After this axiom, an axiom 4a is introduced: Intervals are measurable sets.

Carathéodory, in the same manner as his predecessors, defines an inner measure m_* by setting $m_*(A) = m^*(B) - m^*(B\text{-}A)$ for A subset of B. We note that the inner measure is defined in terms of the outer measure, as was the case in Lebesgue's work. The possibility of defining an inner measure independent of an outer measure was presented by Arthur Rosenthal in his *Beiträge zu Carathéodory Meßbarkeitstheorie*, (Rosenthal, 1916). He presented an axiomatic definition of inner measure entirely independent of outer measure and also defined outer measure in an axiomatic way that turned out to be equivalent to Carathéodory's.

In August 1914 Carathéodory began to write his book *Vorlesungen über reelle Funktionen*, (Carathéodory, 1918) that was published in 1918 but according to a note from the author himself was finished by November 1915. In the preface to this book Carathéodory states that he read the works of Vallée Poussin, Jordan, Baire, Hausdorff, Lindelöf, Young and Lebesgue and the book presents itself as the culmination of the development in the theory that had started at the turn of the Century and as the beginning of the modern axiomatization of this branch of mathematics.

On July 2, 1916 Carathéodory himself described his book in a letter to Féjer in the following way:

> "*I have used the past year to write a book (unfortunately too thick) about real functions which is now being printed. It is no great deed but I believe that it will be useful, since Lebesgue's theory of integration will be presented in an easily understandable manner for the first time*."[6]

This book turned out to be of fundamental significance to both Carathéodory and mathematical analysis, it was one of the reasons why Carathéodory was proposed as a corresponding member of the Mathematical-Physical Class of the Götingen Academy of Sciences in 1919.

In the Introduction Carathéodory states that

> "*The revolution the theory of real functions underwent as a result of the investigations of Mr. Lebesgue constitutes an event that today may be considered to be largely complete. Therefore an attempt to rebuild the theory from its foundations, and systematically, seemed mandatory to me.*" (Carathéodory, 1918, p. v)

Carathéodory defines outer measure $m^*(A)$ as the infimum of the (finite or countable) sums of q-dimensional volumes of the *intervals* that cover A and notes that this is a particular case of the general notion given by the axioms above.

One of the points that we consider crucial to the further development of the theory is the explicit example given by Carathéodory of another set function that satisfies the axioms: $\delta_a(A) = \begin{cases} 1 \text{ si } a \in A \\ 0 \text{ si } \notin A \end{cases}$ where $a \in \mathbf{R}^q$ is fixed and $A \subseteq \mathbf{R}^q$.

[6]The original letter is in German, our translation has been taken from (Georgiadou, 2004, p. 109).

We believe that this represented a turning point in measure theory as the existence of different measures meant that measure theory was now an independent field of study within mathematics with is own object of study: measures.

In the case of Lebesgue's theory the measure in question had always been one that generalised the concepts of length, area, volume, etc. and it was this example given by Carathéodory that transformed the subject and made the goal of the theory to study measures in themselves and not just the sets on which the measures were defined. In this way measure theory became an abstract and general theory; even if it had been presented in a pre-axiomatic way since Borel, it is with Carathéodory and the publication of (Carathéodory, 1918)} that it became a theory on its own with its own objects.

Carathéodory's axiomatization of measure theory.

Now, having stressed the importance of Carathéodory's 1918 text, it is relevant to note that it was republished in 1927 and soon afterwards Teubner requested a third printing but Carathéodory felt that by then the theory of real functions had changed considerably since 1918 and decided to modify his book accordingly before it was published again. In particular he wanted to include abstract spaces that were first introduced by Fréchet as the notion of integral had been extended to these spaces and in this way Lebesgue's theory, that had constituted the central topic of (Carathéodory, 1918) had to be incorporated to these more general theories. Carathéodory decided to take all this into account when preparing the new edition of his book. However, once this work began, Carathéodory discovered that these generalizations could be taken even further as what was really important were the properties that one required of the integral and not the elements over which integration was to be extended. In this way, Carathéodory presented a new series of works:

- *Entwurf für eine Algebraisierung des Integralbegriffs*, 1938
- *Bemerkungen zur Axiomatik der Somentheorie*, 1938
- *Die Homomorphien von Somen und die Multiplikation von Inhaltsfunktionen*, 1939
- *Über die Differentiation von Massfunktionen*, 1940
- *Bemerkungen zum Riesz-Fischerschen Satz und zur Ergodentheorie*, 1941
- *Gepaarte Mengen, Verbände, Somenringe*, 1942
- *Bemerkungen zum Ergodensatz von G. Birkhoff*, 1944.[7]

To include the ideas that began to develop in this new series Carathéodory decided to rewrite his book from 1918 in three volumes. The first volume *Reelle Funktionen, Bd. I, Zahlen, Punktmengen Funktionen*, (Carathéodory, 1939b), was published in 1939 and included all the material from his book prior to the concept of measure.

The publication, in 1938, of *Entwurf für eine Algebraisierung des Integralbegriffs* marked the beginning of a new axiomatization of measure theory. Carathéodory sent the manuscript to Arthur Rosenthal who commented that the general and axiomatic definition

[7] See (Carathéodory, 1938a, 1938b, 1939a, 1940, 1941, 1942 ans 1944) respectively.

of sets struck him as "very amusing and, of course, surely new" and that the system of axioms seemed "natural and comprehensible".[8]

Of the two pending volumes of the Theory of Real Functions that had been formally announced by Carathéodory in the preface to *Reelle Funktionen I*, the second volume was to appear in 1943, however, Teubner was bombed and destroyed during the bombing of Leipzig and Carathéodory decided to revise the contents again.

On March 24, 1949, in a letter addressed to Born, Carathéodory announced the following:

> "*I also have another book on measure and integral in Boolean spaces. It had already been printed but was destroyed by fire in Leipzig in 1943. I have rewritten it. It will also be published by Birkhäuser.*" (Carathéodory, 1949)

After the war Carathéodory went over the material of the second and third volumes and decided to write a single volume that was to be self-contained and covering all the material. However, Carathéodory died a year later on February 2, 1950 and it was Rosenthal, Steuerwald and Finsler who edited the volume and published it posthumously in 1956 as *Mass und Integral und ihre Algebraisierung*.

This text constitutes a unified and systematic presentation of the general theory that Carathéodory began to develop in 1938. The concepts in this text are concepts that have been constructed in an ad hoc fashion for the establishment of a general theory of measure. That is, a theory that contains as particular examples not only the Euclidean theory but also measure theory as it developed in the XX Century.

To achieve this unification new objects were needed; these objects were to have the properties of both sets with arbitrary elements and figures of elemental geometry that cannot be treated as sets of points. These objects were called *somas* by Carathéodory. They were first introduced in (Carathéodory, 1938a) by an axiomatic definition.

To study the final theory presented by Carathéodory in (Carathéodory, 1956) we first introduce the axioms that Carathéodory introduces in his 1938 papers, (Carathéodory 1938a and b). The three sets of axioms are different and the evolution of the theory in Carathéodory's mind becomes evident upon their comparison as does his goal of a completely algebraic theory.

The axioms in (Carathéodory, 1938a) are the following:

[Axiom 1, 1938a] All somas A, B,... form a set $\mathfrak{M}$. For any two somas it is always possible to know if $A = B$ or $A \neq B$. The equality sign here expresses any relation that satisfies the following conditions, $A = A$, if $A = B$ then $B = A$ and if $A = B$ and $B = C$ then $A = C$.

[Axiom 2, 1938a] To each pair of somas A, B a third soma $A \dotplus B$ will be clearly assigned and will be called the union of A and B. The following rules are applied for the union: $A \dotplus A = A,\ \ A \dotplus B = B \dotplus A$ and $B = B'$ implies that $A \dotplus B = A \dotplus B'$.

[8] These quotations are taken from a letter from Rosenthal to Carathéodory dated February 20, 1938 that can be found in *Teilnachlass Carathéodory*.

After this second axiom Carathéodory defines what it means for a soma to be part of another soma: $B \subseteq A$ if $A \dotplus B = A$ and on this basis introduced the third axiom.
[Axiom 3, 1938a] Given a countable sequence of somas A_1, A_2,… there exists a minimal containing soma V that contains all somas in the sequence. This soma will be called their union and we write $V = A_1 \dotplus A_2 \dotplus$ ....
[Axiom 4, 1938a] There exists at least one soma O, the empty soma, that is a part of all somas.

Given the last axiom it is now possible to define what it means for two somas to be disjoint: two somas A and B are disjoint, $A \circ B$, if the only soma that is part of both is the empty soma.
[Axiom 5, 1938a] If a soma B is disjoint from all somas A_1, A_2, … then it is also disjoint from their union V.

Finally, the last axiom establishes the difference between two somas in the following manner:
[Axiom 6, 1938a] If A and B are any two somas, there is always at least one soma B_1 that satisfies the following conditions simultaneously:

$$B_1 \circ A, B_1 \dotplus B = B \text{ and } B_1 \dotplus A = B \dotplus A\,.$$

Carathéodory wrote (Carathéodory, 1938b) a few months later and the sixth section of this article, which in fact is only 9 pages long, is dedicated to “The axioms of the theory of somas”. As we have already noted, these axioms differ slightly from those presented in (Carathéodory, 1938a).

In this section Carathéodory defines a set $\mathfrak{M}_0$ whose elements are the somas A,B, ... of the theory. After this he defines what is means for a soma to be contained in another soma and on this basis proves the following theorem:
Theorem. If $A \subseteq B$ and B⊆A then $A = B$.

This is the first point that comes to our attention when comparing both texts from 1938 as in the former the equality between somas is defined by the symbol = and all that is required is that this symbol be an equivalence relation. In the latter text, as we have just seen, the equality of somas is a theorem derived from the notion of containment.

After introducing this theorem Carathéodory begins section 7 of his article in which he explicitly presents the new axioms for his theory. The first of these is announced by saying that “the existence of the union of somas is determined by the following axiom”:
[Axiom 1, 1938b] Given a finite or countable sequence of somas A_1, A_2, … there exists a soma V such that $A_j \subseteq V$ for all j and such that if for some soma B, $A_j \subseteq B$ holds for all j, then $V \subseteq B$.
The union of somas is defined from this axiom: the union of somas A_1, A_2, … is the soma V and we write $V = A_1 \dotplus A_2 \dotplus$ .....

The second axioms guarantees the existence of the empty soma and is presented after two theorems relative to the union of somas.
[Axiom 2, 1938b] There exists at least one soma O, the empty soma, that is a part of any soma.

On this basis the notion of disjoint somas is defined and finally axioms 3 and 4 are introduced and coincide with axioms 5 and 6 of (Carathéodory, 1938a).

This work laid down the foundations for Carathéodory's seminal work on this topic which we present in the next section.

Carathéodory's algebraization of measure theory.

Carathéodory began his book, *Mass und Integral und ihre Algebraisierung*, by stating new and highly ambitious objectives:

> "*George Boole (1815-1862), in his famous book on logic, Laws of Thought, published in 1854, developed a symbolism that today is called Boolean algebra. The simplest example of such an algebra is that obtained by applying to sets the operations of forming unions, intersections, and differences (or passage from a set to its complement).*
> *It is clear from this that the theory of measure, which can be developed even for sets of arbitrary elements, need not lose its significance even for rings of elements of a Boolean algebra.*
> *Some ten years ago, I noticed that is is also possible to construct the analogue, on Boolean rings, of ordinary point functions, which makes possible the algebraization of the integral.*"(Carathéodory, 1956, p. 5)

It is important to that this goal is not only of theoretical interest but that the theorems and methods of proof that are to be developed show certain relations between objects that would otherwise have remained in the dark. Furthermore, these relations led to a development in the theory that Carathéodory caracterises as "organic, highly elementary and unified". It is in this context that the concept of soma is strengthened.

The first chapter of (Carathéodory, 1956) is dedicated to the study of somas but begins with a brief exposition of the axiomatic method and Carathéodory announces that his intention is to treat the theories of measure and integration within the framework of this method. As we have noted above both Borel and Lebesgue presented their theories in this way in an implicit form, however, Carathéodory aims to go much further not only via an axiomatization but by turning this theory into a purely algebraic one.

Carathéodory presents yet another group of axioms for somas and states that the totality of somas that occur within any given problem will always be a set that will be denoted, in general, by $\mathfrak{M}_0$.

[Axiom 1] Let a collection of somas *A*, *B*,... be given forming a non-empty set $\mathfrak{M}_0$.

To every pair *A*, *B* of somas of $\mathfrak{M}_0$ let there be assigned a third soma of $\mathfrak{M}_0$, denoted by $A \dotdiv B$, and called the conjunction of *A* and *B*. This operation satisfies $A \dotdiv B = B \dotdiv A$ and $A \dotdiv (B \dotdiv C) = (A \dotdiv B) \dotdiv C$ where the equality sign indicates that the two somas are identical, i.e., that the symbols on both sides stand for one and the same soma. Furthermore, for every pair *A*, *B* of somas of $\mathfrak{M}_0$, there is at least one some *X* of $\mathfrak{M}_0$ for which $A \dotdiv X = B$.

The operation $A \dot{-} B$ turns $\mathfrak{M}_0$ into an Abelian group; this axiom was deliberately stated to fulfill this job.

This next Theorem can be proved from Axiom 1:

Theorem. There is precisely one soma *O*, the empty soma, for which the equation $X \dot{-} O = X$ holds identically for all *X* belonging to $\mathfrak{M}_0$. Furthermore, the equation $A \dot{-} X = A$ has for any arbitrary *A* only the one solution *X* = *O*.

After the proof of this Theorem Carathéodory introduces the next axiom:

[Axiom 2] To each pair *A*, *B* of somas in $\mathfrak{M}_0$ −in that order− there is assigned uniquely a third soma of $\mathfrak{M}_0$ denoted by *AB*. For any three somas *A*, *B* and *C*, the following relations hold: $A(BC) = (AB)C$, $(A \dot{-} B)C = AC \dot{-} BC$ and $C(A \dot{-} B) = CA \dot{-} CB$.

Now, any system of somas *A*, *B*, ... for which axioms 1 and 2 hold is a ring. Carathéodory notes that it is not an Abelian ring, nevertheless it does satisfy *AO* = *OA* = *O* for any $A \in \mathfrak{M}_0$.

[Axiom 3.] Every soma A satisfies *AA* = *A*.

That is, the ring that these axioms are posited to define is a Boolean ring. And using this axiom it can be shown that the operation defined by the second axiom is in fact commutative.

Theorem. For any somas *A* and *B* the equations $AB = BA,\ AO = O,\ A \dot{-} A = O$ hold and, furthermore, $A \dot{-} X = B$ implies $X = A \dot{-} B$.

Once Carathéodory has shown that *AB* = *BA* for any two somas he defines the intersection of *A* and *B* precisely as *AB*.

Before introducing the fourth and final axiom, Carathéodory presents the following definitions:

Definition. A soma *A* is called a subsoma of the soma *B* if *AB* = *A*. To express that *A* is a subsoma of *B* we write $A \subseteq B$ or $B \supseteq A$.

Definition. A set $\mathfrak{M}_0$ is called partially ordered if a relation $A \subseteq B$ is defined for certain pairs of elements *A*, *B* of $\mathfrak{M}_0$ in such a way that both $A \subseteq A$ for all $A \in \mathfrak{M}_0$ and if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The following theorem, regarding the union of somas, is proved following these definitions:

Theorem. Any two somas *A* and *B* have a (unique) minimal containing soma, called their union and denoted by $A \dot{+} B$. The union can be defined by $A \dot{+} B = A \dot{-} B \dot{-} AB$.

And the concept of union can be extended to infinite collections:

Definition. We say that $V = \sum_{A \in \mathfrak{A}} A$ is the union of the somas *A* of some set $\mathfrak{A}$ of somas if *V* satisfies that if $A \in \mathfrak{A}$ then $A \subseteq V$, and if $A \subseteq W$ for all $A \in \mathfrak{A}$ then $V \subseteq W$.

These two conditions in the definition guarantee that if a union of somas exists then it is unique and it can be shown that the union of a finite collection of somas always exists. However, the fact that the union of a countable collection of somas exists needs an axiom:

[Axiom 4.] Every sequence A_1, A_2, ... of countably many somas has a minimal containing soma called the union of the A_j and written $A_1 \dot{+} A_2 \dot{+} \ldots$ or $\sum_j A_j$.

This axiom does not follow from the previous ones and it is important to note that it doesn't imply that the union of an arbitrary collection of somas always exists. To show this, Carathéodory takes as somas the sets of Lebesgue measure zero in **R**, the sets that consist of a single point in [0,1] constitute a set of somas but their union (the whole interval) is not one of the somas as it does not have measure zero.

It is on the basis of these axioms that Carathéodory is to develop his theory. It is very important to note that the axioms are propositions that deal with these new objects: somas; and not measure as one could have imagined when comparing with existing measure theories.

With these axioms the algebraic theory of somas is established to then introduce the concept of soma functions and further on that of measurable function.

Definition. A soma U in the domain of definition $\mathfrak{A}$ of a soma function $F(X)$ is said to be F-measurable, if for every soma A which, together with AU and $A \dot{-} AU$, belongs to $\mathfrak{A}$ and for which in addition the numbers $F(A)$, $F(AU)$ and $F(A \dot{-} AU)$ are finite, the equality $F(A) = F(AU) + F(A \dot{-} AU)$ always holds.

And we note that for a soma U to be F-measurable it is not necessary for $F(U)$ to be finite. Now, to introduce the concept of measure Carathéodory needs the following definitions:

Definition. A set of somas $\mathfrak{A}$ is called additive if whenever $A \in \mathfrak{A}$ and $B \in \mathfrak{A}$, the union $A \dot{+} B$ of these two somas is also an element of $\mathfrak{A}$. Analogously, a set of somas $\mathfrak{A}$ is called multiplicative, conjunctive, and subtractive, according as $A \in \mathfrak{A}$ and $B \in \mathfrak{A}$ together imply $AB \in \mathfrak{A}$, $A \dot{-} B \in \mathfrak{A}$ and $A - AB \in \mathfrak{A}$.

A set of somas $\mathfrak{R}$ which is additive, multiplicative, conjunctive and subtractive is called a ring of somas.

Carathéodory notes that another terminology is available for these objects. He notes that Hausdorff, in his *Mengenlehre* of 1914, calls a set that is closed under the four basic binary operations a field. However, if the operations on point sets are considered as Boolean operations then the notions of field and ring coincide. Carathéodory also notes that in his previous publications he assumed for his treatment of measure and the integral the usages of set theory and hence, used the term field of somas; nevertheless the development of lattice theory and Boolean algebras made him change his terminology. The following definition is also an example of this change in terminology, Hausdorff called the concept Carathéodory is defining a σ-field.

Definition. A ring of somas which is at the same time a countably additive set of somas is called a complete ring.

Finally, Carathéodory arrives at the crucial point for which the previous theory has been developed:

Definition. A soma function $\phi(X)$ is called a measure function whenever it satisfies the following conditions:

1. its domain of definition $\mathfrak{A}$ is a complete ring;
2. if whenever an at most countable number of somas A_j of $\mathfrak{A}$ and any other soma B of $\mathfrak{A}$ satisfy the relation $B \subseteq A_1 \dot{+} A_2 \dot{+} ...$, then the relation $\phi(B) \leq \phi(A_1) + \phi(A_2) + \ldots$ also holds;
3. for the empty soma O we have $\phi(O)=0$.

These conditions can be used to show that $\phi(X)$ is nonnegative for all somas X.

This definition in which it is easy to recognize both Lebesgue's measure as the modern notion of measure has been constructed by Carathéodory in an axiomatic manner as we have shown starting from new objects in such a way that it has become a purely algebraic notion that can be applied to the elements of any Boolean algebra. This in turn allows the introduction of a theory of integration which culminates Carathéodory's theory and in which the integral is defined by means of the following theorem:

Theorem. Let $\phi(X)$ be a measure function and f a ϕ-measurable, non-negative place function, whose domain of definition M is a normal soma of the ring of measurability $f(X)$. Then there is one and only one measure function $\psi(X)$ that is cometric to $\phi(X)$ and absolutely continuous with respect to $\phi(X)$, that is defined for all subsomas of M, and that satisfies the relations $\alpha(A)\phi(\mathrm{A}) \leq \psi(A) \leq \beta(A)\phi(\mathrm{A})$ for all those subsomas A of M for which $0 < \phi(A) < \infty$. Here $\alpha(A)$ and $\beta(A)$ denote the infimum and the supremum of f on the soma A.

The measure function $\psi(X)$ is called the integral of f on X for the measure function $f(X)$ and is denoted by $\psi(X) = \int f d\phi$.

This theory of integration has a particular case the Lebesgue integral which also is built upon a theory of measure. This is interesting as there are may cases of integrals defined during the twentieth century whose foundations lie on ideas similar to the ones used by Riemann to define his integral and not on a theory of measure. This, however, will be the topic of a second paper on the development of integrals that generalise the Lebesgue integral. What is in fact important for this paper is to note the algebraic aspect of Carathéodory's contribution to the theory of integration.

In (Kappos, 1974, p. 253), for instance, Demetrios Kappos notes that since probability is a normed measure on a Boolean algebra of events "then the algebraic measure theory of Carathéodory is very suitable to introduce the concept of probability as a strictly positive and normed measure."

Probability Theory: Halmos and Kolmogorov.

Andrei Kolmogorov's *Grundbegriffe der Wahrscheinlichkeitsrechnung* of 1933 is well known as the symbol of modern probability theory as it laid out the axiomatic foundations of the theory; however, it will not be object of our discussion here. We would like to focus on a paper written by Paul Halmos in 1944, *The Foundations of Probability* and a paper published by Kolmogorov in 1948, *Complete metric Boolean Algebras*. Kolmogorov's paper was originally given as a lecture in Russian at the VI Congress of Polish Mathematicians and published in French as *Algèbres de Boole mètriques complètes*, (Kolmogorv, 1948).

Halmos, (Halmos, 1944, p. 493), begins his paper by stating that "probability is a branch of mathematics" and then states very clearly what the intention of the paper is:

> "*The purpose of this paper is exposition, exposition intended to convince the professional mathematician that probability is mathematics.*"

In order to achieve this goal Halmos intends to give at least a partial answer to the question “What is probability?” It is the answer to this question that we find relevant for our present topic of study.

After posing this question Halmos dedicates the second section of his text to Boolean Algebras and concludes that:

> “*The mathematical theory of probability consists of the study of Boolean σ-algebras.This is not to say that all Boolean σ-algebras are within the domain of probability theory. In general statements concerning such algebras and the relations between their elements are merely qualitative: probability theory differs from the general theory in that it studies also the quantitative aspects of Boolean algebras.*” (Halmos, 1944, p. 496)

Halmos then goes on to introduce measure spaces and claims that it can be shown that the theories of measure and probability are coextensive:

> “*If B is any Boolean σ-algebra and P a probability measure on B, then there exists a measure space Ω such that the system B is abstractly identical with an algebra of subsets of Ωreduced by identification according to sets of measure zero, and the value of P for any event a is identical with the values of the measure for the corresponding subsets of Ω.*” (Halmos, 1944, p. 499)

Kolmogorov presents these ideas very clearly and rigorously. He starts by defining the notion of a metric Boolean Algebra. To do this he calls the unit element of a Boolean algebra *u* and the null element *n*, a metric Boolean algebra can then be defined as follows:

Definition.A metric Boolean Algebra is defined as a Boolean algebra together with a real-valued function *m* of the elements of the algebra, where *m*(*x*), called the measure of the element *x*, is defined for every *x* in the algebra and satisfies the postulates:

1. $x \cap y = n$ implies $m(x \cup y) = m(x) + m(y)$,
2. $x \neq n$ implies $m(x) > 0$.

The element complementary to x will be designated by x' and the relation of inclusion $x \subseteq y$ will be understood in the sense of $x \cup y = y$. If $x \circ y$ then denotes the symmetric difference, a distance ρ can be introduced between elements of the metric Boolean algebra (justifying thus the name): $\rho(x,y) = m(x \circ y)$.

This distance turns the algebra into a metric space, and if the metric space is complete the metric Boolean algebra will also be said to be complete. Kolmogorov then justifies why this is important:

> “*The general definition of metric Boolean algebra which we have now seen is a natural generalization of properties of a series of classical mathematical entities which have long been known. So, e.g., when the formal definitions are properly chosen, the regions of an arbitrary*

> *bounded domain of space, with their volumes designated as their measures,*
> *form a metric Boolean algebra. Another important example of metric Boolean algebras is provided by the systems of events in an arbitrary problem in probability theory, with their respective probabilities designated as measures.*"(Kolmogorov, 1948, p. 59)

Kolmogorov then states that the classical notions of volume of a figure and probability of an event are most commonly conceived as measures of sets. Due to this fact the theory of metric Boolean algebras and the general theory of measure can be considered as "parallel and concurrent formal, logical treatments of their concrete subject-matter, which is the same for both theories."(Kolmogorov, 1948, p. 62)

To characterise this formal and logical correlation Kolmogorov uses the notion of isomorphism between metric Boolean algebras:

Definition. Two metric Boolean algebras are isomorphic when there exists a biunique correspondence $x^* = f(x)$, $x = f^{-1}(x^*)$ between the sets of their elements for which we have identically: $f(x \cap y) = f(x) \cap f(y); f(x \cup y) = f(x) \cup f(y)$ and $m\big(f(x)\big) = m(x)$.

Kolmogorov considers measures $m(x)$ which are real-valued, non-negative, completely additive functions defined on a Borel field F_m of subsets of a certain fundamental set U_m which itself belongs to F_m. He then assigns the sets belonging to F_m to disjoint classes or metric types: Two sets X and Y belong to the same metric type when $m(X \circ Y) = 0$.

As all sets of the same metric type have the same measure it is natural to regard it as the measure of the metric type itself, and then the following results can be proved:

Proposition 1. The metric types of any measure form a complete metric Boolean algebra.

Proposition 2. Every complete metric Boolean algebra is isomorphic to the algebra of metric types of some measure.

Two measures will be called structurally isomorphic when the algebras of their metric types are isomorphic. Now, given propositions 1 and 2 Kolmogorov announces that the theory of complete metric Boolean algebras is equivalent to the theory of measures, considered up to a structural isomorphism. And this is important for the theory of probability as the complete algebra of events can always be transformed isomorphically into the algebra of the metric types of a suitably constructed measures.

The last section of this paper by Kolmogorov is then dedicated to the classification of complete metric Boolean algebras.

> "*The passage from the theory of measures to complete metric Boolean algebras has a highly significant consequence: The complete metric Boolean algebras can be (up to an isomorphism) perfectly surveyed and classified.*"(Kolmogorov, 1948, p. 64)

To present this classification, Kolmogorov introduces the following concepts: Let the set $S(x)$ designate the set of all elements $y \subseteq x$, then the weight $\tau(x)$ of x will be the smallest of the powers of sets dense in $S(x)$ and the element x will be called homogeneous when the power of every element $y \neq \emptyset$ in $S(x)$ is equal to that of the element x.

It can be shown that $\tau(x)$ in a complete metric Boolean algebra is always equal to 1, 2 or an infinite power; and in the case that it is 2, x will be called an atom.

The classification is then presented as the following theorem due Dorothy Maharam published in (Maharam, 1942): The unit element u of any complete metric Boolean algebra can be represented in the form $u = (\bigcup_r a_r) \cup (\bigcup_s c_s)$ with $r = 1, 2, \ldots$; $s = 1, 2, \ldots$ Where a_r are atoms and the c_s are homogeneous elements of infinite weight, and these elements satisfy two further conditions:

1. $m(a_1) \geq m(a_2) \geq \cdots$
2. $\tau(c_1) < \tau(c_2) < \cdots$

The unions in the right-hand side of the decomposition can be finite or denumerable. The decomposition satisfying (1) and (2) is unique (up to a permutation of atoms of equal measure). In order that two complete metric Boolean algebras be isomorphic, it is necessary and sufficient that the sequences (of real and transfinite numbers)

$$I \begin{cases} m(a_1), m(a_2), \ldots \\ \tau(c_1), \tau(c_2), \ldots \\ m(c_1), m(c_2), \ldots \end{cases}$$

coincide for the one algebra and for the other.

A final remark.

We would like to conclude our present discussion by noting that \linebreak Carathéodory's algebraization of the integral did not immediately lead to an algebraic integration theory. In fact, it was Irving Segal in (Segal, 1965) who introduced the term and actually felt a need to explain it:

> "*It was the development of a variety of new theories, rather than the desire to embellish old ones, which primarily has led to the development of a complex of results, methods, and ideas here somewhat loosely referred to as `algebraic integration theory'. The introduction of a new term such as this requires some explanation and justification, in the light of the rapidly increasing burden which mathematicians must bear*
> [...] *This might be called invariant integration theory, but this term might well suggest a subject quite different from this one, namely that of invariant integrals. Since the setting of the theory is naturally algebraic, in its concern with features independent of isomorphisms, the term algebraic integration theory is reasonable -although the subject is distinctly more distant from conventional algebra than is algebraic topology. Such a theory is necessarily abstract, but the term `abstract integration theory' has already a different meaning, signifying usually the*

theory in which integrals are considered not necessarily over subsets of euclidean space, but over relatively general spaces, and is a more limited and quite distinct notion from that of the theory considered here, whose distinctive description as algebraic seems therefore practical."

Bibliography.

BANACH, Stefan. 1923. Sur le problème de la mesure, *Fundamenta Mathematica*, 4, 7-33.
BOREL, Emile. 1898. *Leçons sur la Théorie des Fonctions*, Paris.
CANTOR, Georg. 1884. Ueberunendlichen, lineare Punktmannichfaltigkeiten 5, *Math. Annalen*, 23, 453-488.
CARATHÉODORY, Constantin. 1914. Überdaslineare Mass von Punktmengen – eine Verallgemeinerung des Längenbegriffs, *Nach. der K. Gesellschaft derWiss. Zu Göttingen. Math.-phys. Klasse*, 404-426.
CARATHÉODORY, Constantin. 1918. *Vorlesungen über ReelleFunktionen*.
CARATHÉODORY, Constantin. 1938a. Entwurffüreine Algebraisierung des Integralbegriffs, *S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss.*, 27-69.
CARATHÉODORY, Constantin. 1938b. Bemerkungen zur Axiomatik der Somentheorie, *S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss.*, 175-183.
CARATHÉODORY, Constantin. 1939a. Die Homomorphien von Somen und die Multiplikation von Inhaltsfunktionen, *Ann. Scuola Norm. Sup. Pisa*, 2, Vol. 8, 105-130.
CARATHÉODORY, Constantin. 1939b. *Reelle Funktionen, Bd. I, Zahlen, Punktmengen Funktionen*, Teubner, Leipzig.
CARATHÉODORY, Constantin. 1940. Über die Differentiation von Massfunctionen, *Math. Z.*, Vol. 46, 181-189.
CARATHÉODORY, Constantin. 1941. BemerkungenzumRiesz-FischerschenSatzund zurErgodentheorie, *Abh. Math. Sem. Hansischen Universität*, Vol. 14, 351-389.
CARATHÉODORY, Constantin. 1942. Gepaarte Mengen, Verbände, Somenringe, *Math. Z.*, Vol. 48, 4-26.
CARATHÉODORY, Constantin. 1944. Bemerkungen zum Ergodensatz von G. Birkhoff, *S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss*., 189-208.
CARATHÉODORY, Constantin. 1949. Letter from Carathéodory to Born, 24 March 1949, Staatsbibliothek zu Berlin, Preussischer Kulturbesitz, Handschriftenabteilung, Nachlass Born 105.
CARATHÉODORY, Constantin. 1956. *Mass und Integral und ihre Algebraisierung*, Birkhäuser.
CARATHÉODORY, Constanin. *Teilnachlass Carathéodory*, Bayer, Akad. d. Wiss.
GEORGIADOU, Maria. 2004. *Constantin Carathéodory. Mathematics and Politics in Turbulent Times*, Springer Verlag, Germnay.
HADAMARD, Jacques. 1905. Cinq lettres sur la théorie des ensembles, *Bulletin de la Société Mathématique de France*, 33, 261 - 273.

HALMOS, Paul. 1944. The Foundations of Probability, *American Mathematical Monthly*, 51, 493-510.
HARNACK, Adolf v. 1880. Über die trigonometrische Reihe und die Darstellungwillkürlicher Functionen, *Math. Annalen*, 17, 122-32.
HAUSDORFF, Felix. 1914. *Grundzüge der Mengenlehre*.
HAWKINS, Thomas. 1975. *Lebesgue's theory of integration*, Chelsea Publishing Company, New York.
JORDAN, Camille. 1892. Remarques sur les intégrales définies, *Journal de Mathématiques pures et appliquées* 4, 8, 69-99.
KAPPOS, Demetrios. 1974. Generalized Probability with Applications to Quantum Mechanics, in (Panayotopoulus, 1974), 253-270.
KOLMOGOROV, Andrei. 1933. *Grundbegriffeder Wahrscheinlichkeitsrechnun*, Springer, Berlin.
KOLMOGOROV, Andrei. 1948. Algèbres de Boole mètriques complètes, *VI Zjazd Matematyków Polskich*, Warsaw, 21-30, 1948. Translated into English: Complete metric Boolean Algebras, *Philosophical Studies*, 77, 57-66, 1995.
LEBESGUE, Henri. 1902. Intégrale, longueur, aire, *Annali di Mathematica*, III-7, 31-129.
MAHARAM, Dorothy. 1942. On homogeneous measure algebras, *Proc. Nat. Acad. Sci.*, 28, 108-111.
MARTÍNEZ ADAME, Carmen. 2013 ¿Es necesario el Axioma de Zermelo para comprender la teoría de la medida? *Metatheoria*, 3, (2), 37-64.
MICHEL, Alain. 1992. *Constitution de la théorie moderne de l'intégration*, Vrin, Francia.
MOORE, Gregory. 1982. *Zermelo's Axiom of Choice, Its Origins, Development and Influence*, Springer Verlag, New York.
PANAYOTOPOULOS, A. (ed) 1974. *C. Caratheodory International Symposium 1973*, Proceedings Greek Mathematical Society, Athens, 1974.
PEANO, Guiseppe. 1883. Sulla integrabilità delle funzioni, *Atti della R. Accademia delle Scienze di Torino*, 18, 439-446.
RASSIAS, Themistocles. (ed.) 1991. *Constantin Carathéodory. An international tribute*, 2 vols., World Scientific Publ. Co., Singapore.
ROSENTHAL, Arthur. 1916. Beiträge zu Carathéodory Messbarkeitstheorie, *Nachr. Ges. Wiss. Göttingen*, 305-321.
SEGAL, Irving. 1951. Equivalences of Measure Spaces, *American Journal of Mathematics*, 73, 2, 275-313.
SEGAL, Irving. 1965. Algebraic Integration Theory, *Bull. Amer. Math. Soc*., 71, 3, Part 1, 419-489.
STOLZ, Otto. 1884. Ueber einen zu einer unendlichen Punktmenge gehörigen Grenzwert, *Math. Annalen* 23, 152-156.
VITALI, Guiseppe. 1905. *Sul problema della misura dei gruppi di punti di una retta*, Bologna, Tip. Gamberini e Parmeggiani.

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